Chapter 6
Time and Frequency Characterization of Signals and Systems

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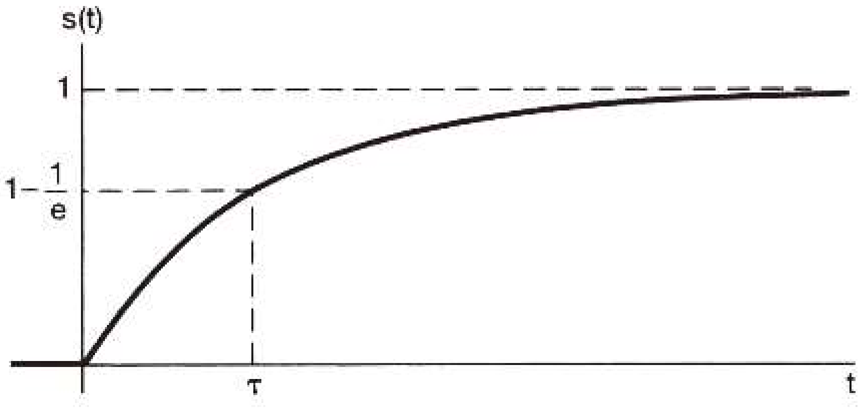
6.0 Introduction

The frequency-domain characterization of an LTI system in terms of its frequency response represents an alternative to the time-domain characterization through convolution. In analyzing LTI systems, it is often particularly convenient to utilize the Frequency domain because **differential and difference equations and convolution operations in the time domain become algebraic operations in the frequency domain**.

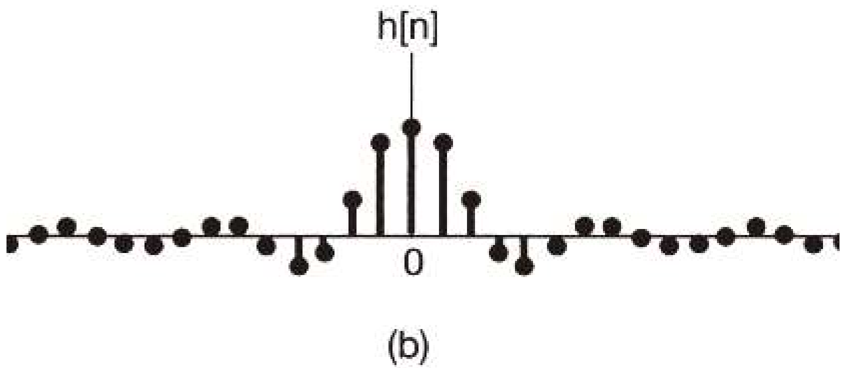
6.0 Introduction

Moreover, concept such as frequency-selective filter are readily understood in frequency domain.

However, when designing a system, analyzing both time and frequency domain is typically required.



Unit step response



Causality or oscillation

6.1 The Magnitude-Phase Representation of The Fourier Transform

The magnitude-phase representation of the continuous-time Fourier transform $X(j\omega)$ is

$$X(j\omega) = |X(j\omega)| e^{j\measuredangle X(j\omega)}. \tag{6.1}$$

連續時間傅立葉轉換的大小—相位表示法

Similarly the magnitude-phase representation of the discrete-time Fourier transform $X(e^{j\omega})$ is

$$X(e^{j\omega}) = |X(e^{j\omega})| e^{j\measuredangle X(e^{j\omega})}. \tag{6.2}$$

離散時間傅立葉轉換的大小—相位表示法

6.1 The Magnitude-Phase Representation of The Fourier Transform

we can think of *X(jω)* as providing us with a decomposition of the signal *x(t)* into a "sum" of complex exponentials at different frequencies.

*X(jω)*告訴我們訊號*x(t)*可分解成不同頻率的複指數的和。

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega(4.8) \qquad x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega,$$

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt(4.9) \qquad X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x[n]e^{-j\omega n}.(5.9)$$

6.1 The Magnitude-Phase Representation of The Fourier Transform

The magnitude $|X(j\omega)|$ describes the basic frequency **content** of a signal—i.e., $|X(j\omega)|$ provides us with the information about the relative magnitudes of the complex exponentials that make up *x(t)*.

Parseval's Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \boxed{|X(j\omega)|^2} d\omega.$$

Energy-density spectrum

$|X(j\omega)|^2 d\omega / 2\pi$ **Amount of energy at ω**

6.1 The Magnitude-Phase Representation of The Fourier Transform

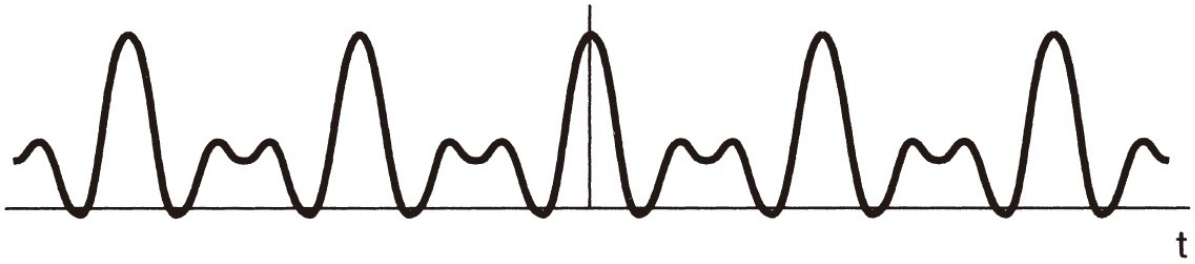
The phase angle $\not\subset X(j\omega)$, on the other hand, does not affect the **amplitudes of the individual frequency components**, but instead provides us with information concerning the **relative phases of these exponentials**.

However, note that even magnitudes haven't changed, due to change in phase angle, the signal in time-domain will change.

6.1 The Magnitude-Phase Representation of The Fourier Transform

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3). \qquad (6.3)$$

In general, changes in the phase function of $X(j\omega)$ lead to changes in the time-domain characteristics of the



(a)

$\varphi_1 = 0$

$\varphi_2 = 0$

$\varphi_3 = 0$

(b)

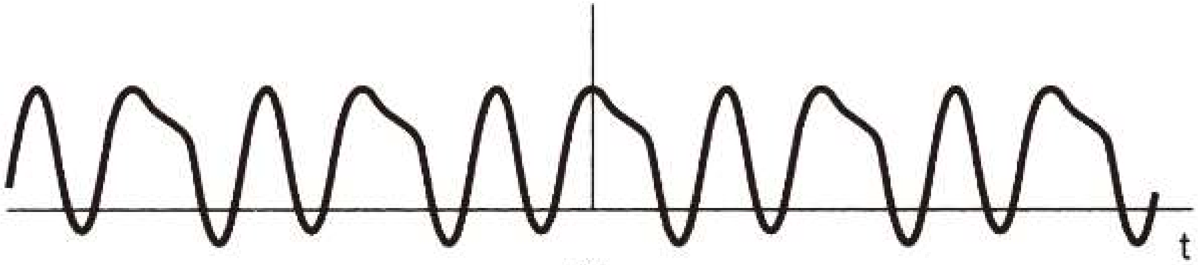
$\varphi_1 = 4$

$\varphi_2 = 8$

$\varphi_3 = 12$

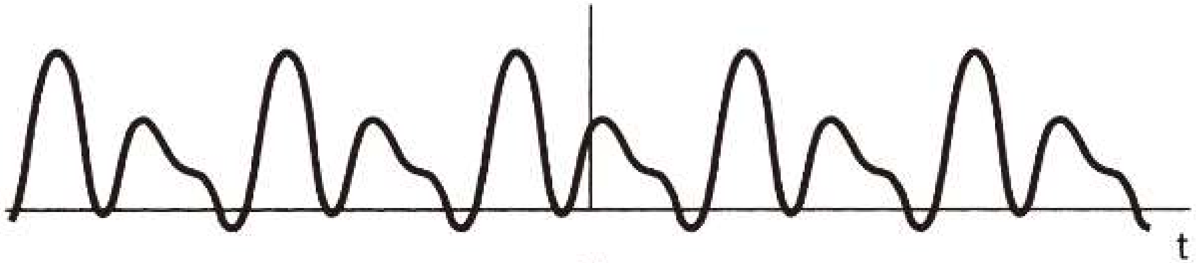
6.1 The Magnitude-Phase Representation of The Fourier Transform

$$x(t) = 1 + \frac{1}{2}\cos(2\pi t + \phi_1) + \cos(4\pi t + \phi_2) + \frac{2}{3}\cos(6\pi t + \phi_3).$$



(c)

$$\varphi_1 = 6$$
$$\varphi_2 = -2.7$$
$$\varphi_3 = 0.93$$



(d)

$$\varphi_1 = 1.2$$
$$\varphi_2 = 4.1$$
$$\varphi_3 = -7.02$$

6.1 The Magnitude-Phase Representation of The Fourier Transform

In some instances **phase distortion** may be important, whereas in others it is not.

Example, time reverse signal:

The corresponding effect in the frequency domain is to replace the Fourier transform phase by its negative:

$$F\{x(-t)\} = X(-j\omega) = |X(j\omega)| e^{-j\measuredangle X(j\omega)}.$$

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

The transform *$Y(j\omega)$* of the output of an LTI system is related to the transform *$X(j\omega)$* of the input to the system by the equation

$$Y(j\omega) = H(j\omega)X(j\omega),$$

連續時間LTI系統的輸出輸入傅立葉轉換和系統頻率響應的關係

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

The Fourier transforms of the input $X(e^{j\omega})$ and output $Y(e^{j\omega})$ of an LTI system with frequency response $H(e^{j\omega})$ are related by

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}). \tag{6.4}$$

The effect that an LTI system has on the input is to change the **complex amplitude** of each of the frequency components of the signal.

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

The nature of the effect in more detail. Specifically, in continuous time,

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)| \tag{6.5}$$

輸入輸出在各頻率的振幅關係

and

$$\sphericalangle Y(j\omega) = \sphericalangle H(j\omega) + \sphericalangle X(j\omega), \tag{6.6}$$

輸入輸出在各頻率的相角關係

6.2 The Magnitude-Phase Representation of The Frequency Response of LTI Systems

For this reason, $|H(j\omega)|$ (or $H(e^{j\omega})$) is commonly referred to as the ***gain*** of the system.

故 $|H(j\omega)|$或 $H(e^{j\omega})$)常稱為系統的「增益」。

and $\sphericalangle H(j\omega)$ is typically referred to as the ***phase shift*** of the system.

$\sphericalangle H(j\omega)$稱為系統的「相位移」。

6.2.1 Linear and Nonlinear Phase

Consider the continuous-time LTI system with frequency response

若頻率響應 $$H(j\omega) = e^{-j\omega t_0}, \qquad (6.7)$$

so that the system has unit gain and **linear** phase—i.e.,

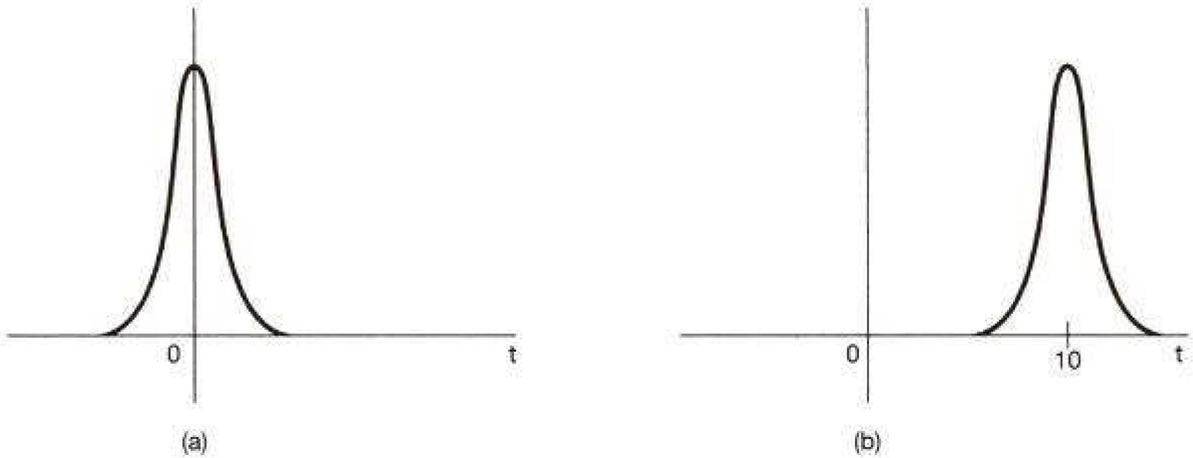
$$|H(j\omega)| = 1, \qquad \measuredangle H(j\omega) = -\omega t_0. \qquad (6.8)$$

則系統具有單位增益及線性相位移。

6.2.1 Linear and Nonlinear Phase

The system with this frequency response characteristic produces an output that is simply a time shift of the input—i.e.,

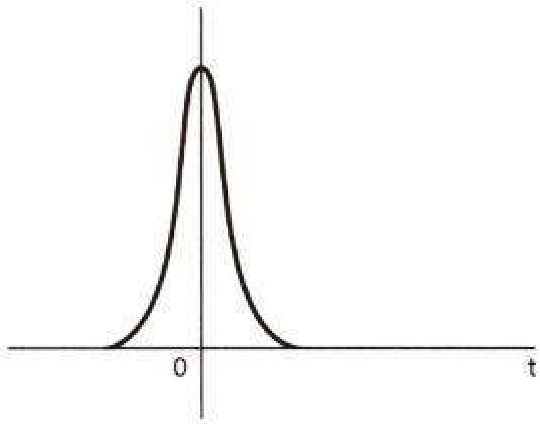
$$y(t) = x(t - t_0). \qquad (6.9)$$



6.2.1 Linear and Nonlinear Phase

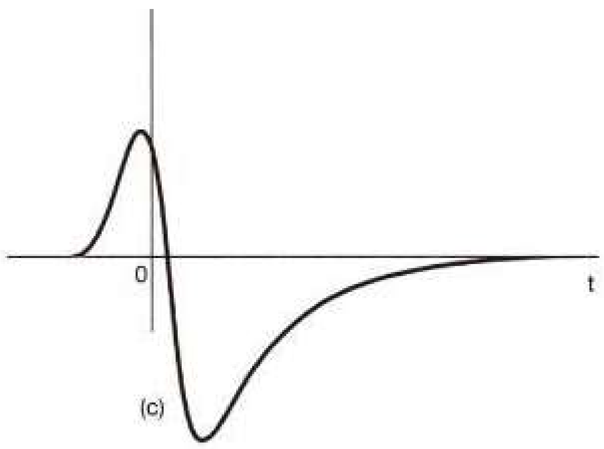
display the output when the signal is applied to a system with unity gain and **nonlinear** phase function—i.e.,

$$H_2(j\omega) = e^{j\angle H_2(j\omega)}, \qquad (6.10)$$



(a)

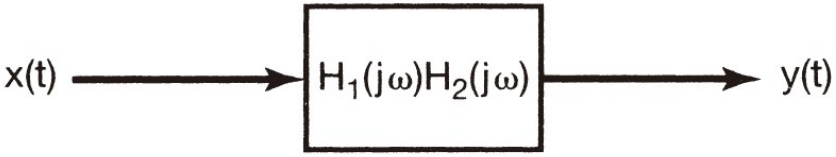
x(t)



y(t)

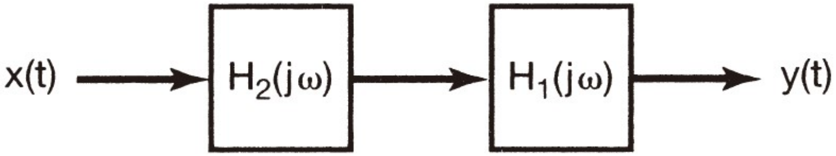
6.2.1 Linear and Nonlinear Phase

In this cas … cy response
has a phas … ling a linear
phase tern



(b)

$$H_3(\quad (6.11)$$



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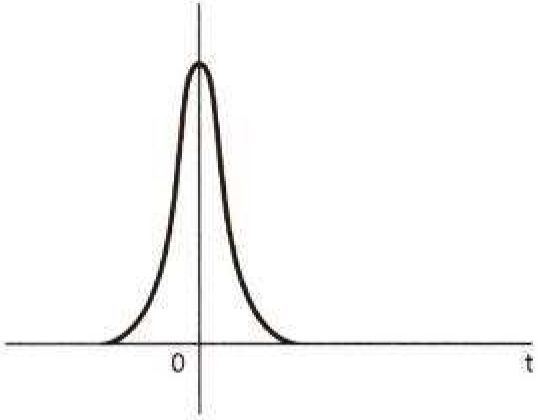
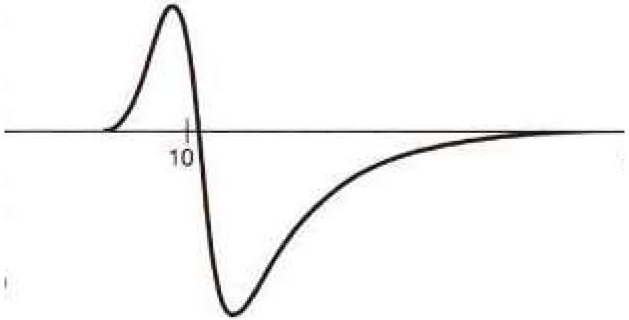
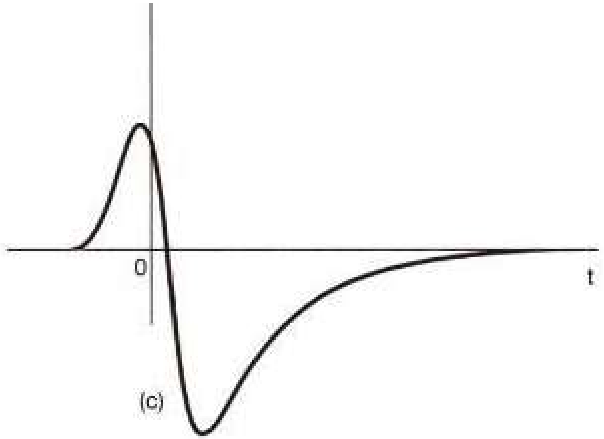


(a)

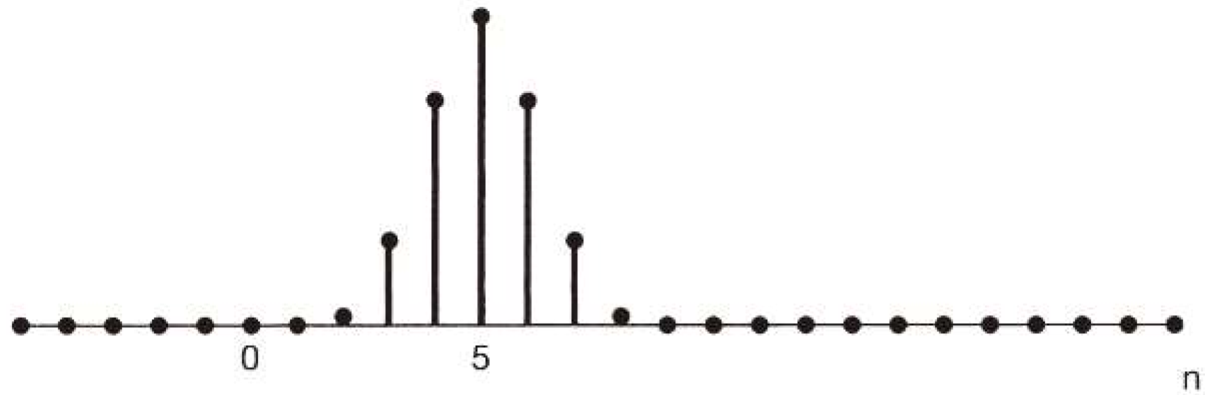




x(t) **y(t)**

6.2.1 Linear and Nonlinear Phase

(a)



(b)

6.2.1 Linear and Nonlinear Phase

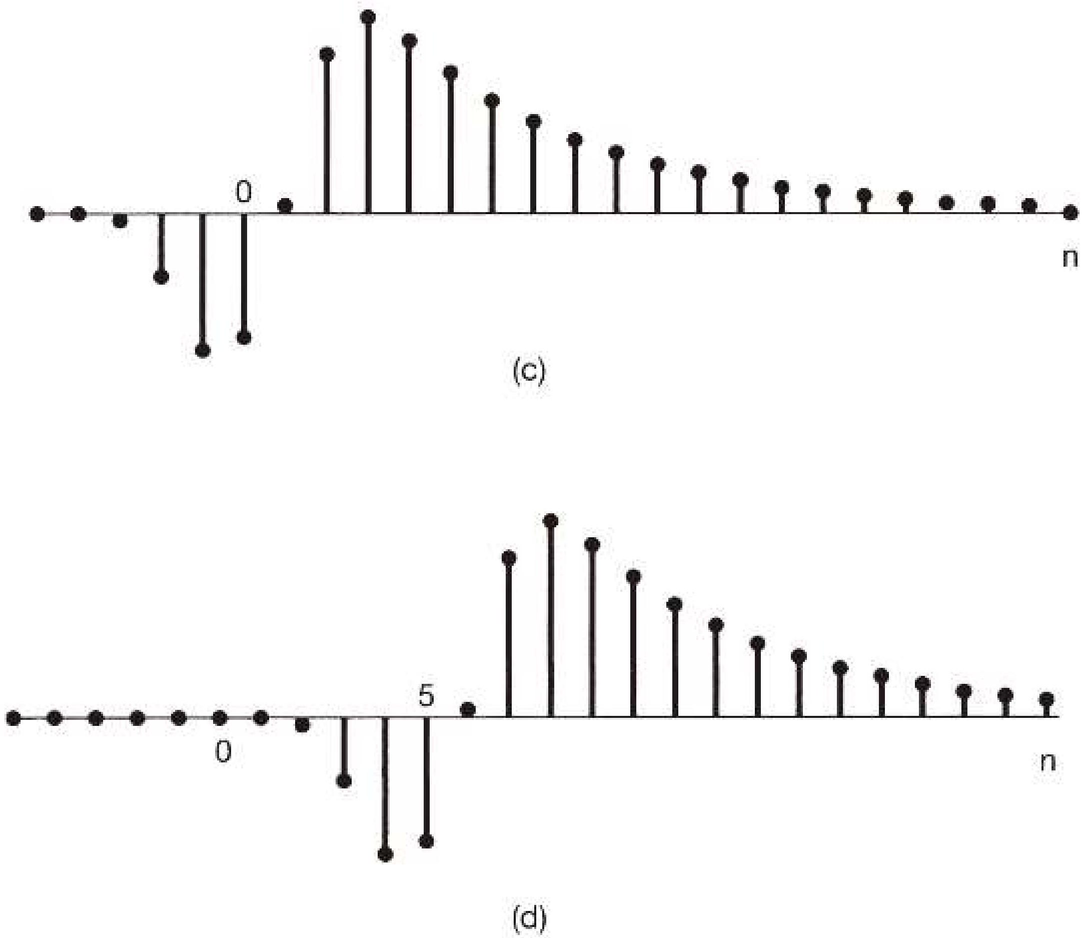


Figure 6.4 (a) Discrete-time signal that is applied as input to several systems for which the frequency response has unity magnitude; (b) response for a system with linear phase with slope of −5; (c) response for a system with nonlinear phase; and (d) response for a system whose phase characteristic is that of part (c) plus a linear phase term with integer slope.

6.2.2 Group Delay

A system with linear phase shift has the simple time-shift interpretation.

$$|H(j\omega)| = 1, \qquad \measuredangle H(j\omega) = -\omega t_0 \quad .$$

$$\Downarrow$$

$$y(t) = x(t - t_0).$$

Time-shift of $-t_0$
or **delay of t_0**

6.2.2 Group Delay

By taking the band to be very small, we can accurately approximate the phase of this system in the band with the linear approximation

$$\measuredangle H(j\omega) \approx -\phi - \omega\alpha, \tag{6.12}$$

so that

$$Y(j\omega) \approx X(j\omega)\left|H(j\omega)\right|e^{-j\phi}e^{-j\omega\alpha}. \tag{6.13}$$

6.2.2 Group Delay

The group delay at each frequency equals the negative of the slope of the phase at that frequency; i.e., the group delay is defined as

$$\tau(\omega) = -\frac{d}{d\omega}\{\measuredangle H(j\omega)\}. \tag{6.14}$$

Example 6.1

Consider the impulse response of an all-pass system with a group delay that varies with frequency. The frequency response *H(jω)* for our example is the product of three factors; i.e.,

$$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega),$$

Example 6.1

Where

$$H_i(j\omega) = \frac{1 + \left(j\omega / \omega_i\right)^2 - 2j\zeta_i\left(\omega / \omega_i\right)}{1 + \left(j\omega / \omega_i\right)^2 + 2j\zeta_i\left(\omega / \omega_i\right)}, \qquad (6.15)$$

$$\begin{cases} \omega_1 = 315 rad/\sec \quad and \zeta_1 = 0.066, \\ \omega_2 = 943 rad/\sec \quad and \zeta_2 = 0.033, \\ \omega_3 = 1888 rad/\sec \quad and \zeta_3 = 0.058. \end{cases}$$

Example 6.1

It is often useful to express the frequencies ω_i measured in radians per second in terms of frequencies f_i measured in Hertz, where

$$\omega_i = 2\pi f_i .$$

In this case,

$$f_i \approx 50Hz$$

$$f_2 \approx 150Hz$$

$$f_3 \approx 300Hz.$$

Example 6.1

$$H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\zeta_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\zeta_i(\omega/\omega_i)}$$

Since the numerator of each of the factors $H_i(j\omega)$ is the complex conjugate of the corresponding denominator, it follows that $|H_i(j\omega)| = 1$. Consequently, we may also conclude that

$$|H(j\omega)| = 1.$$

Example 6.1

$$H_i(j\omega) = \frac{1 + \left(j\omega/\omega_i\right)^2 - 2j\zeta_i\left(\omega/\omega_i\right)}{1 + \left(j\omega/\omega_i\right)^2 + 2j\zeta_i\left(\omega/\omega_i\right)}$$

The phase for each $H_i(j\omega)$ can be determined from eq. (6.15):

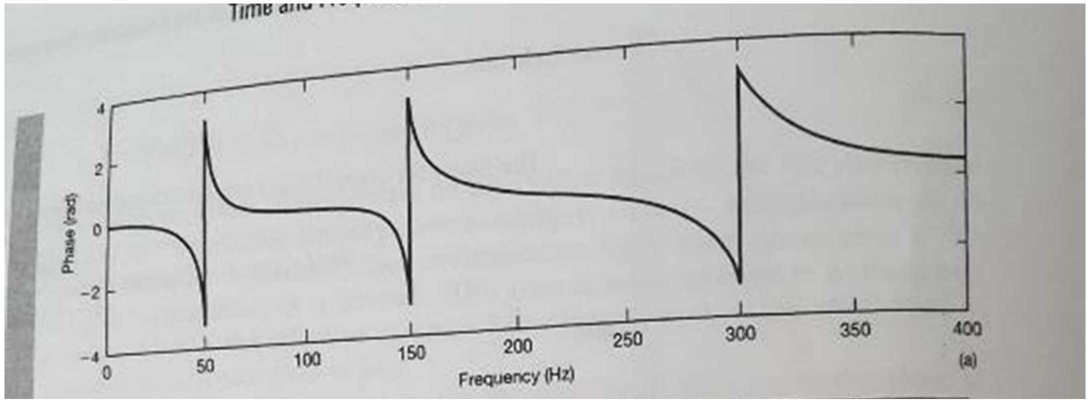
$$\measuredangle H_i(j\omega) = -2\arctan\left[\frac{2\zeta_i\left(\omega/\omega_i\right)}{1 - \left(\omega/\omega_i\right)^2}\right],$$

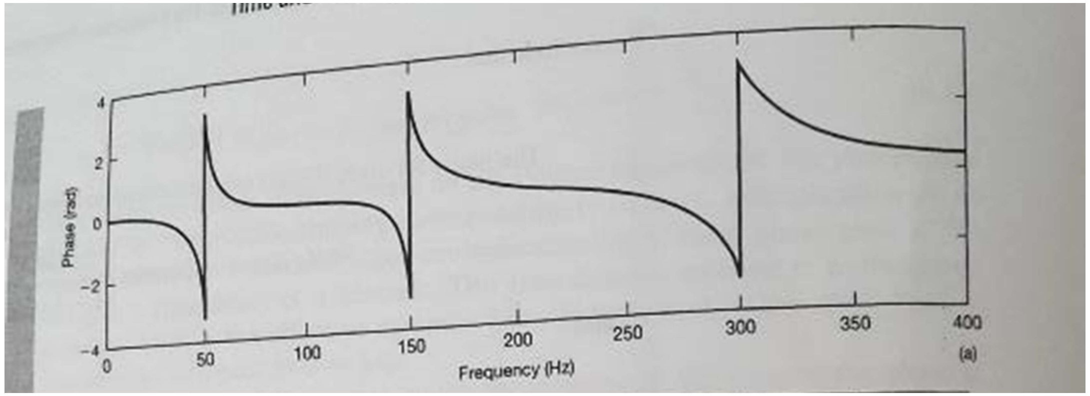
and

$$\measuredangle H(j\omega) = \sum_{i=1}^{3} \measuredangle H_i(j\omega).$$

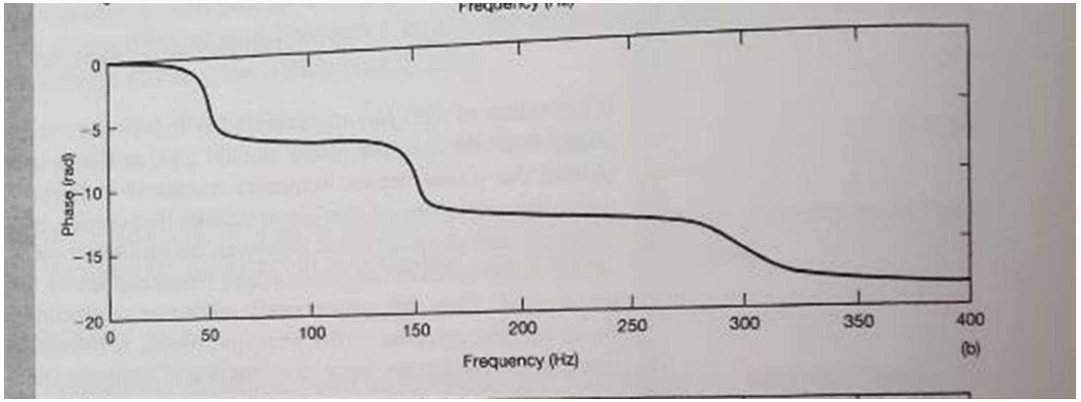
Example 6.1

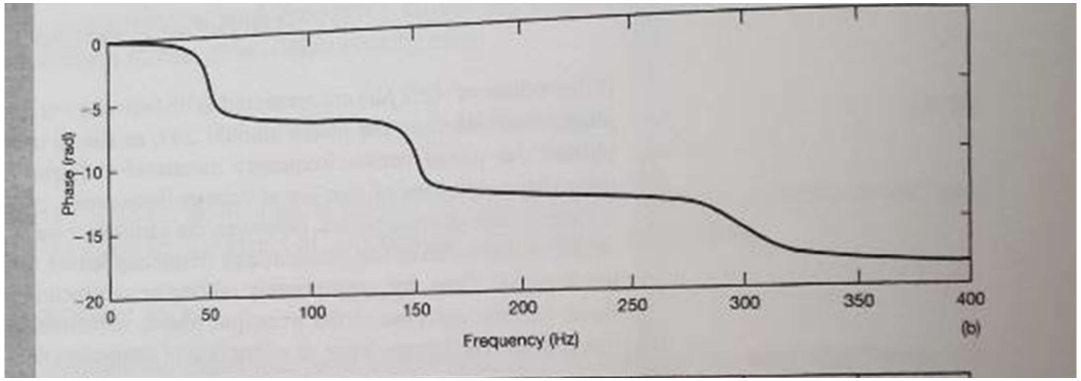
If the values of $\measuredangle H(j\omega)$ are restricted to lie between $-\pi$ and π, we obtain the ***principal-phase*** function (i.e., the phase modulo 2π), as shown in figure 6.5(a) where we have plotted the phase versus frequency measured in Hertz. Note that this function contains **discontinuities** of size 2π at various frequencies, making the phase function non-differentiable at those points.





However, the addition or subtraction of any integer multiple of 2π to the value of the phase at any frequency leaves the original frequency response unchanged. Thus, by appropriately adding or subtracting such integer multiples of 2π from various portions of the principal phase, we obtain the **unwrapped phase** in Figure 6.5(b).

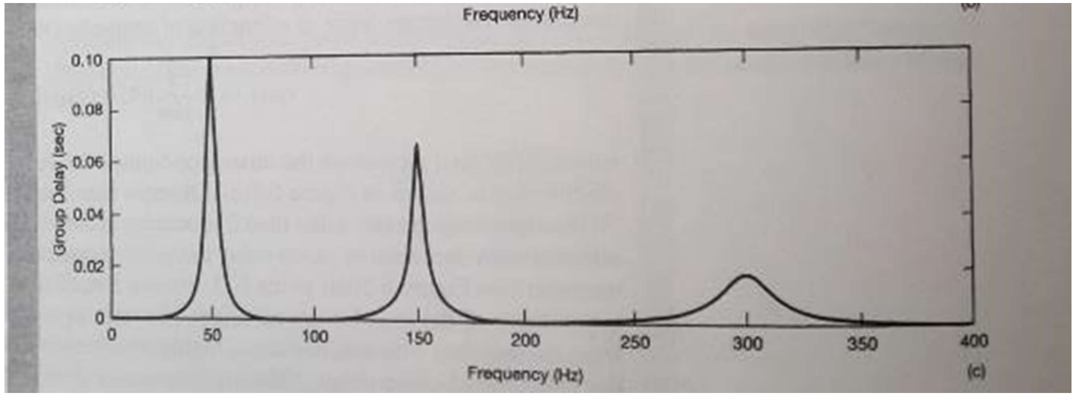




The group delay as a function of frequency may now be computed as

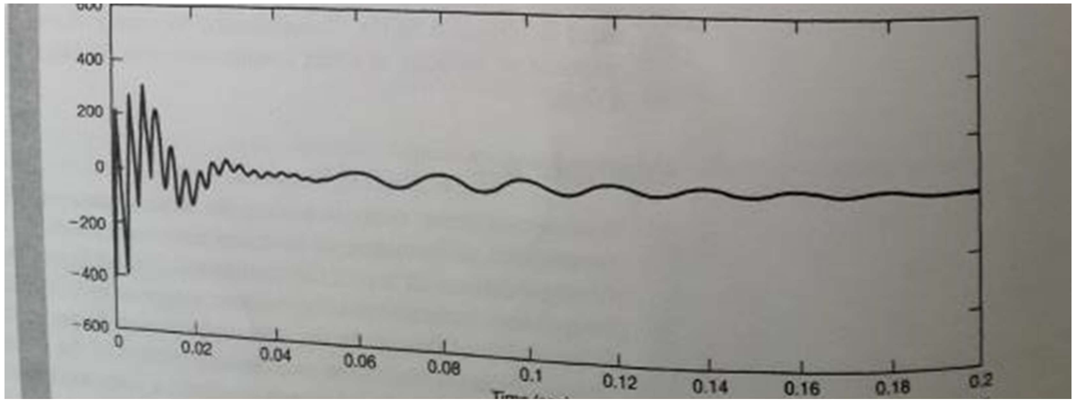
$$\tau(\omega) = -\frac{d}{d\omega}\{\not\subset[H(j\omega)]\},$$

where $\not\subset H(j\omega)$ represents the unwrapped-phase function corresponding to $H(j\omega)$. A plot of $\tau(\omega)$ is shown in Figure 6.5(c).



Example 6.1

Observe that frequencies in the close vicinity of 50 Hz experience greater delay then frequencies in the vicinity of 150 Hz or 300 Hz. The effect of such nonconstant group delay can also be qualitatively observed in the impulse response (see Figure 6.5(d)) of the LTI system.



Example 6.1

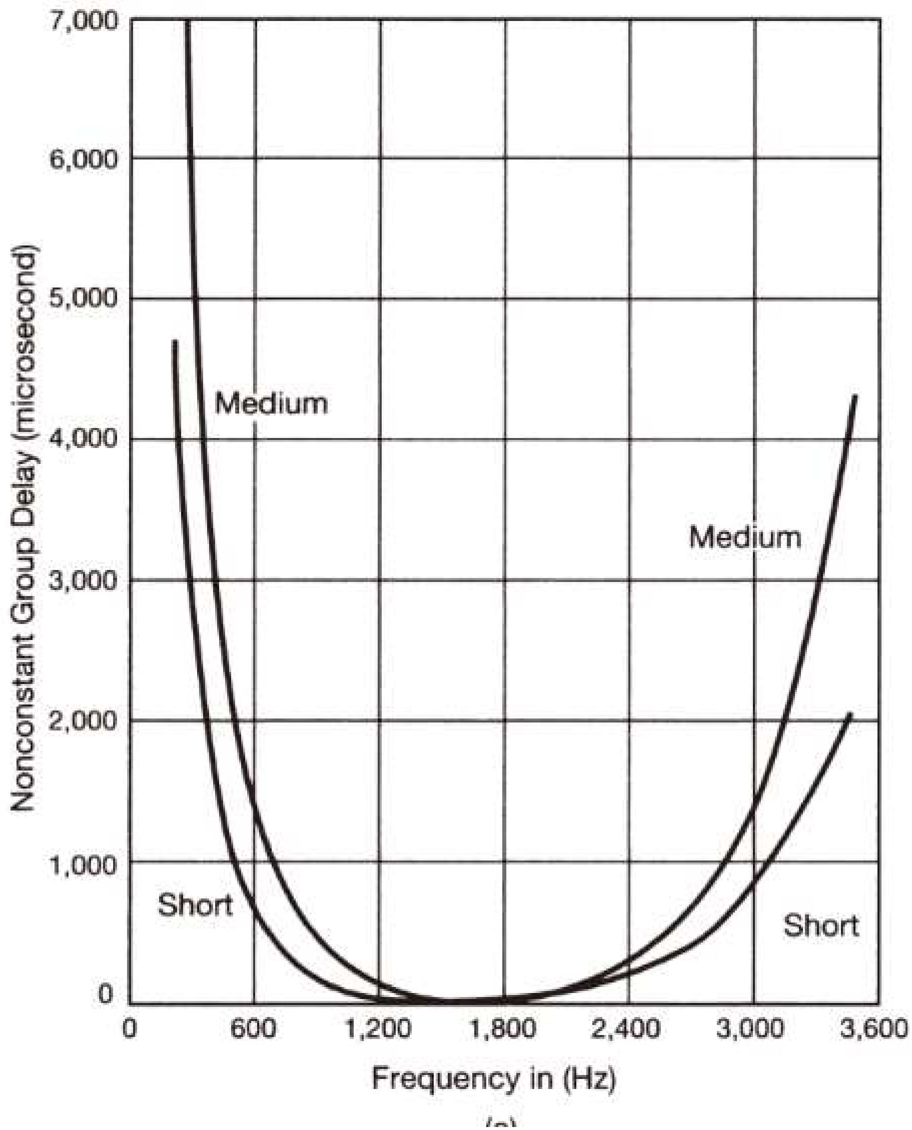
Recall that $F\{\delta(t)\} = 1$. The frequency components of the impulse are **all aligned in time in such a way that they combine to form the impulse**, which is, of course, highly localized in time. Since the all-pass system has nonconstant group delay, different frequencies in the input are delayed by different amounts. This phenomenon is referred to as *dispersion*. In the current example, the group delay is highest at 50 Hz. Consequently, we would expect **the latter parts** of the impulse response to oscillate at lower frequencies near 50Hz. This clearly evident in Figure 6.5(d).

Example 6.2

Nonconstant group delay is among the factors considered important for assessing the transmission performance of switched telecommunications networks. In a survey involving locations all across the continental United States, AT&T/Bell System reported group delay characteristics for various categories of toll calls.

Example 6.2

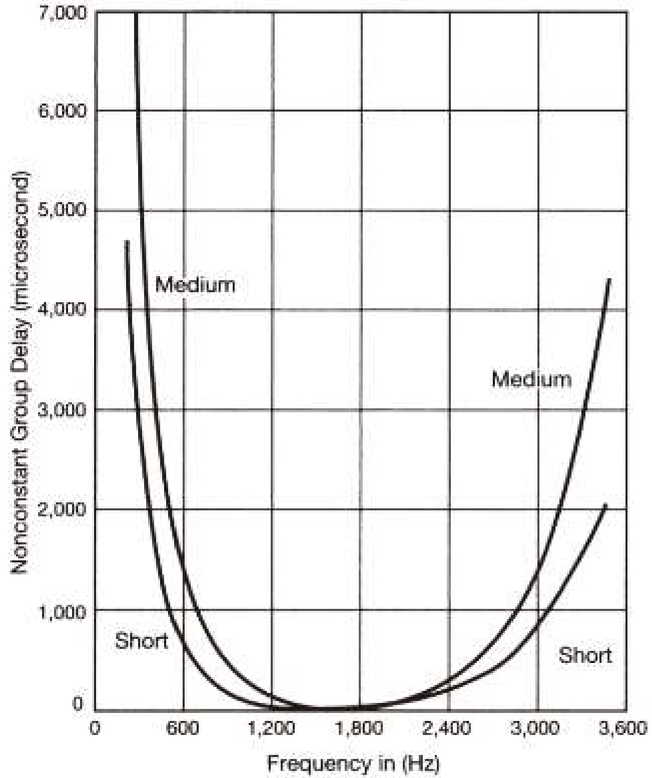
Figure 6.6 displays some of the results of this study for two such classes. In particular, what is plotted in each curve in Figure 6.6(a) is the **nonconstant portion** of the group delay for a specific category of toll calls.

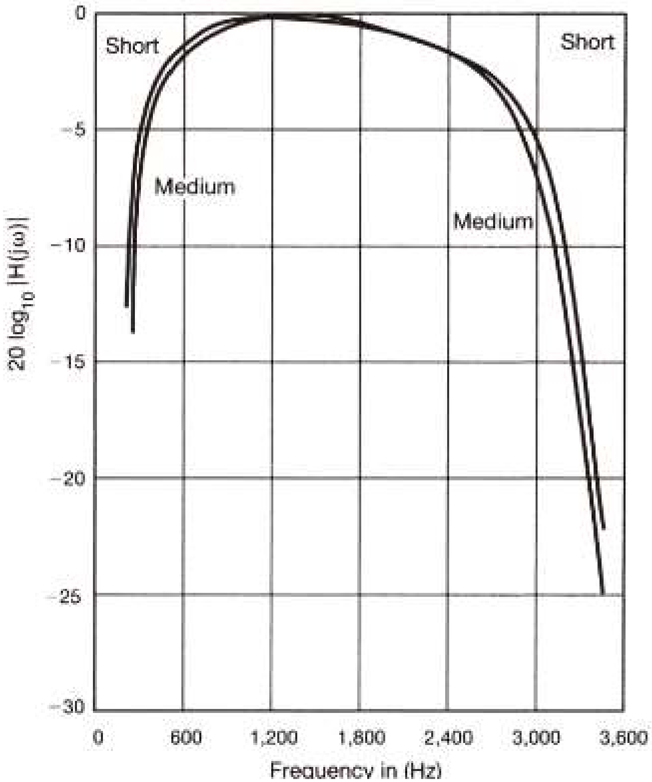


The group delay as a function of frequency is seen to be lowest at 1,700 Hz and increases monotonically as we move away from that figure in either direction.

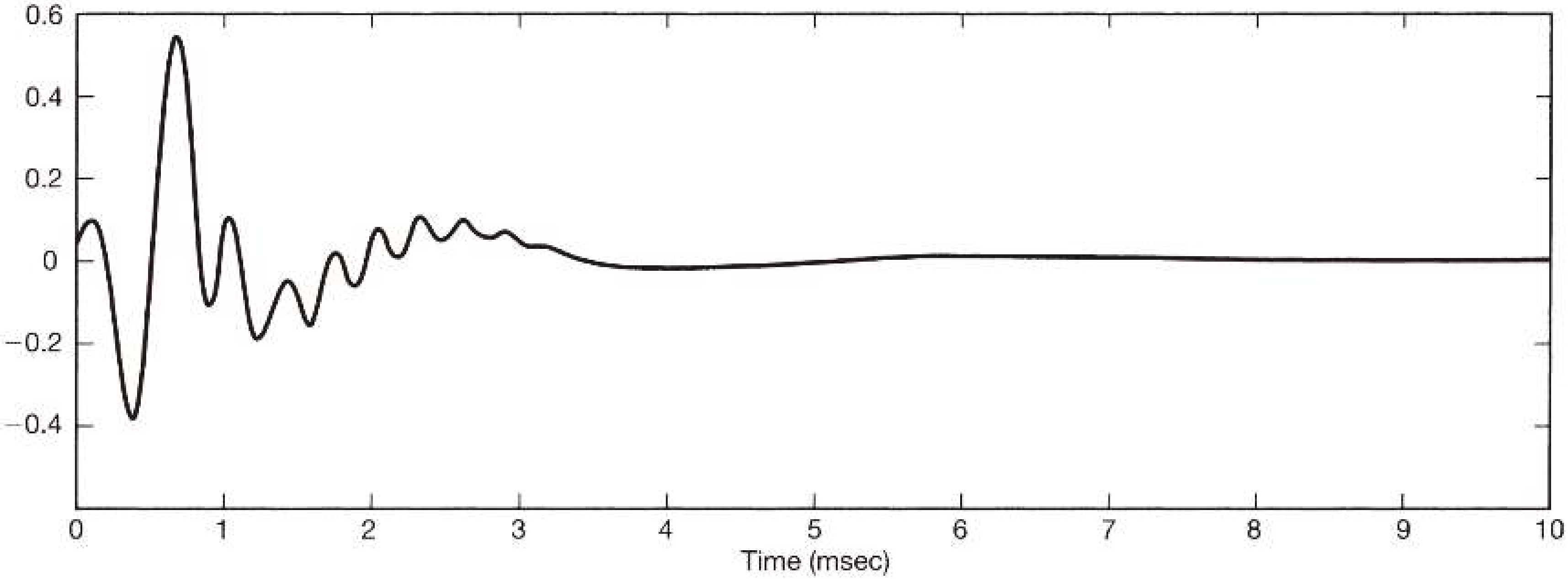
When the group delay characteristics illustrated in Figure 6.6(a) are combined with the characteristics of the magnitude of the frequency response in figure 6.6(b), we obtain impulse reponses of the type shown in Figure 6.7.

Human voice approximately 300 Hz to 3400 Hz





Example 6.2



(a)

The impulse response in Figure 6.7(a) corresponds to the short-distance category. The very low- and very high-frequency components of the response occur later than the components in the mid-frequency range. This is compatible with the corresponding group delay characteristics in Figure 6.6(a).

6.2.3 Log-Magnitude and Phase Plots

In graphically displaying continuous-time or discrete-time Fourier transforms and system frequency responses in polar form, it is often convenient to use a **logarithmic scale** for the magnitude of the Fourier transform.

傅立葉轉換和系統頻率響應，常用對數大小刻度的圖形表示。

6.2.3 Log-Magnitude and Phase Plots

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)| \qquad \sphericalangle Y(j\omega) = \sphericalangle H(j\omega) + \sphericalangle X(j\omega),$$

from eqs. (6.5) and (6.6), which relate the magnitude and phase of the output of an LTI system to those of the input and frequency response. Note that the phase relationship is **additive**, while the magnitude relationship involves the **product** of $|H(j\omega)|$ and $|X(j\omega)|$. Thus, if the magnitudes of the Fourier transform are displayed on **a logarithmic amplitude scale**, eq. (6.5) takes the form of an **additive relationship**, namely,

$$\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|,$$

6.2.3 Log-Magnitude and Phase Plots

For example, on a linear scale, the detailed magnitude characteristics in the stop band of a frequency-selective filter with high attenuation are typically not evident, whereas they are on a logarithmic scale.

The typical log amplitude scale used is in units of $20\log_{10}$, referred to as ***decibels*** (***dB***).

$$0dB =$$

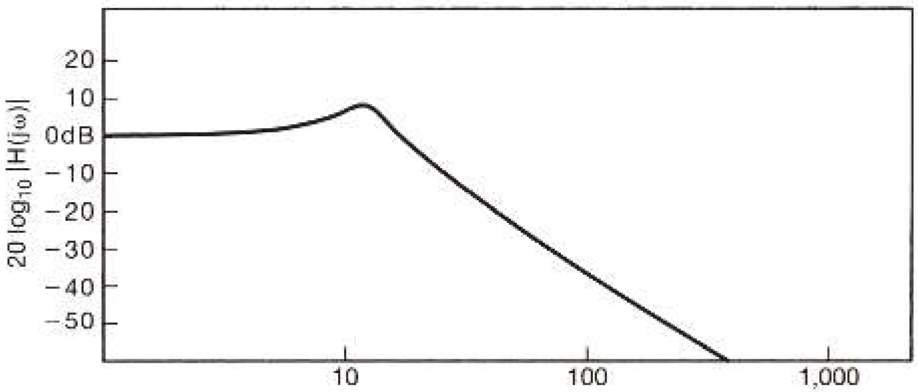
$$20dB = \qquad -20dB =$$

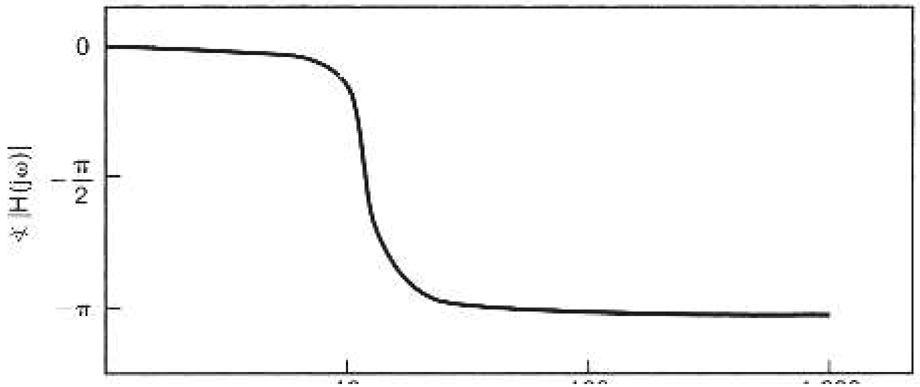
$$6dB \approx$$

6.2.3 Log-Magnitude and Phase Plots

Bode plot

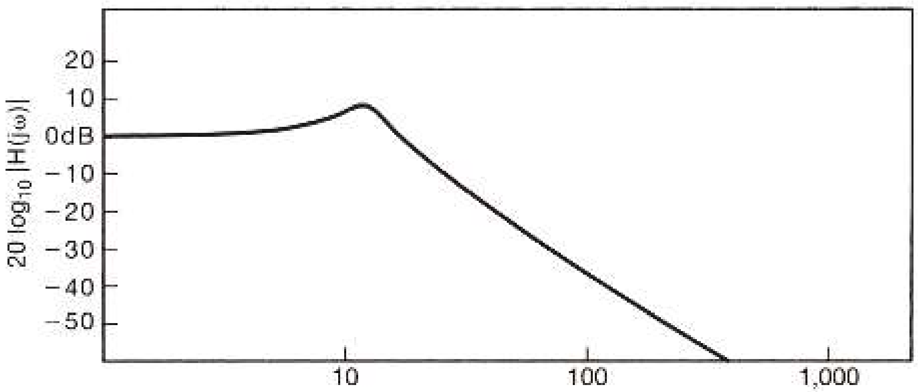
If *h(t)* is real, then $|H(j\omega)|$ is an **even** function of ω and $\measuredangle H(j\omega)$ is an **odd** function of ω. Because of this , the plots for negative ω are superfluous and can be obtained immediately from the plots for positive ω.

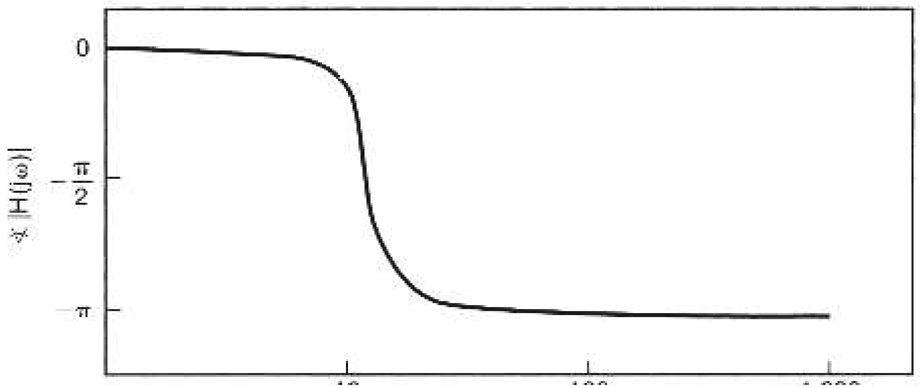




6.2.3 Log-Magnitude and Phase Plots

The use of a **logarithmic frequency scale** offers a number of advantages in continuous time. For example, it often allows a much wider range of frequencies to be displayed than does a linear frequency scale. In addition, on a logarithmic frequency scale, the shape of a particular response curve doesn't change if the frequency is scaled.





6.2.3 Log-Magnitude and Phase Plots

Furthermore for continuous-time LTI systems described by differential equations, an approximate sketch of the log magnitude vs. log frequency can often be easily obtained through the use of asymptotes.

6.2.3 Log-Magnitude and Phase Plots

In discrete time, the **magnitudes** of Fourier transforms and frequency responses are often displayed is dB for the same reasons that they are in continuous time. However, log scale in frequency is not often used since they are periodic with period 2π

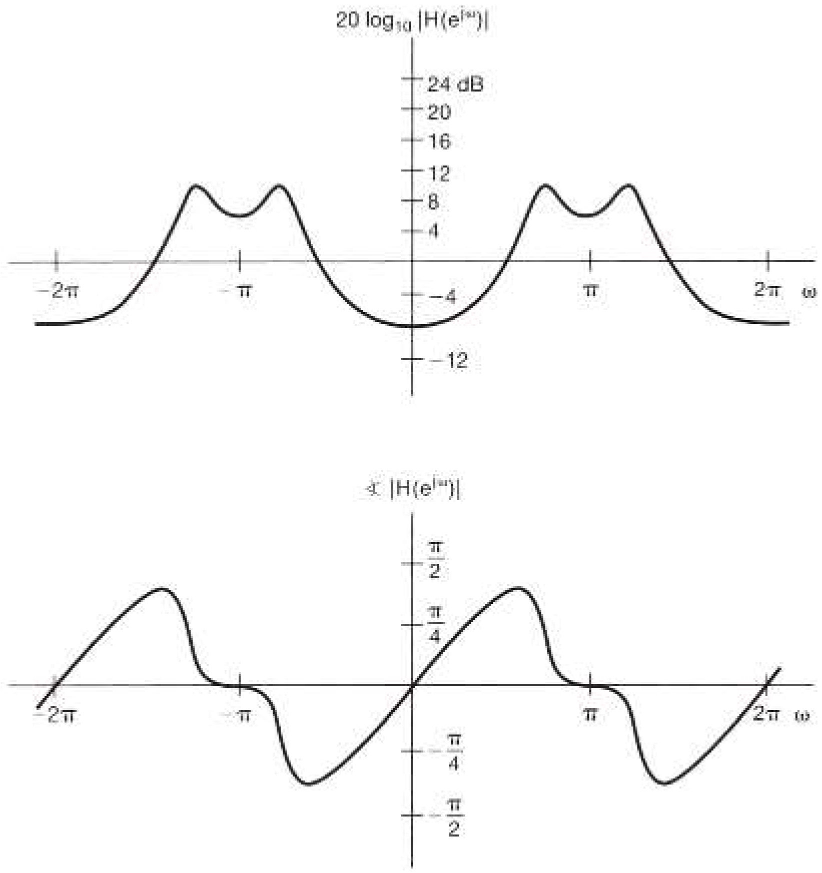


Figure 6.9 Typical graphical representations of the magnitude and phase of a discrete-time frequency response $H(e^{j\omega})$.

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

As introduced in Chapter 3, a continuous-time ideal lowpass filter has a frequency response of the form

$$H(j\omega)=\begin{cases}1 & |\omega|\langle\,\omega_c\\ 0 & |\omega|\rangle\,\omega_c\end{cases}. \tag{6.17}$$

This is illustrated in Figure 6.10(a). Similarly, a discrete-time ideal lowpass filter has a frequency response

$$H(e^{j\omega})=\begin{cases}1 & |\omega|\le\omega_c\\ 0 & \omega_c\langle|\omega|\le\pi\end{cases} \tag{6.18}$$

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

連續時間理想低通濾波器的頻率響應(配合(6.17)式)

離散時間理想低通濾波器的頻率響應(配合(6.18)式)

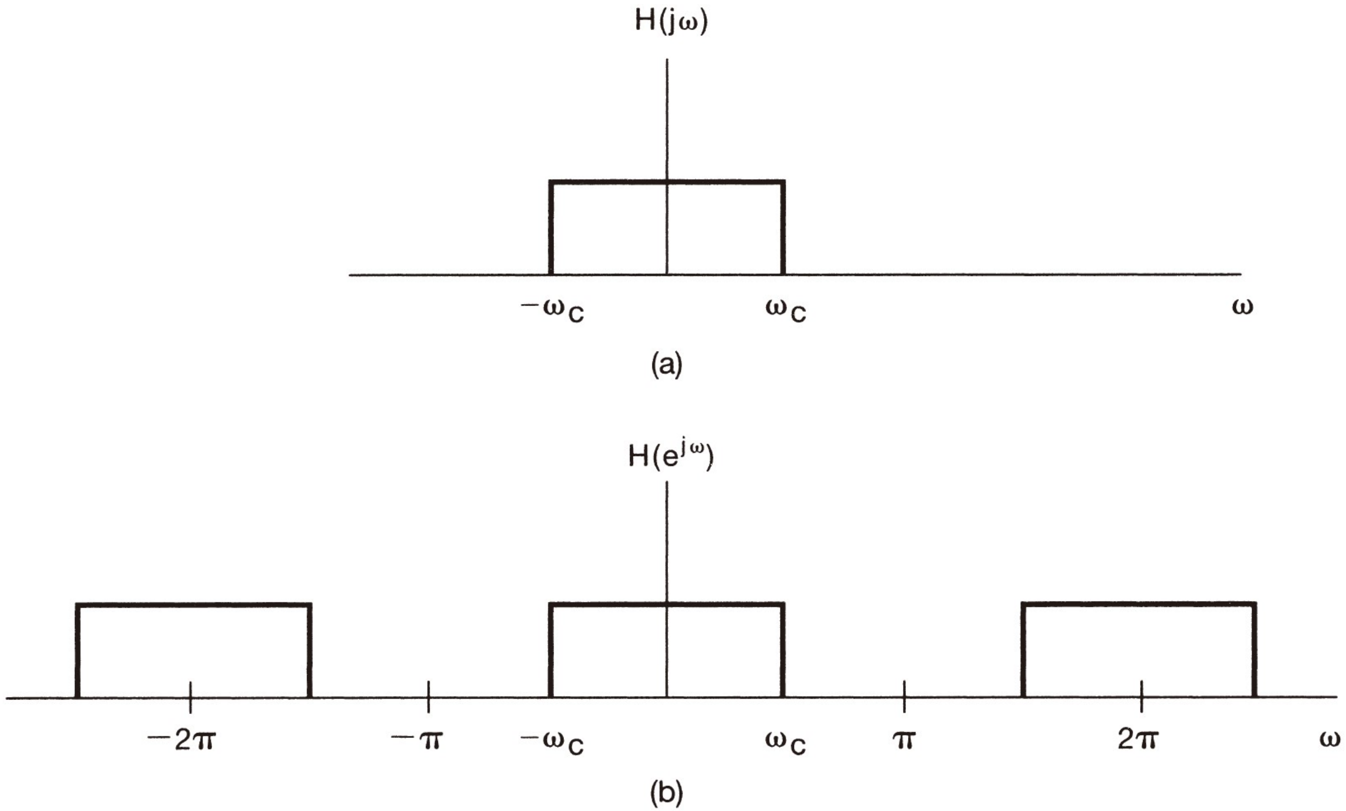
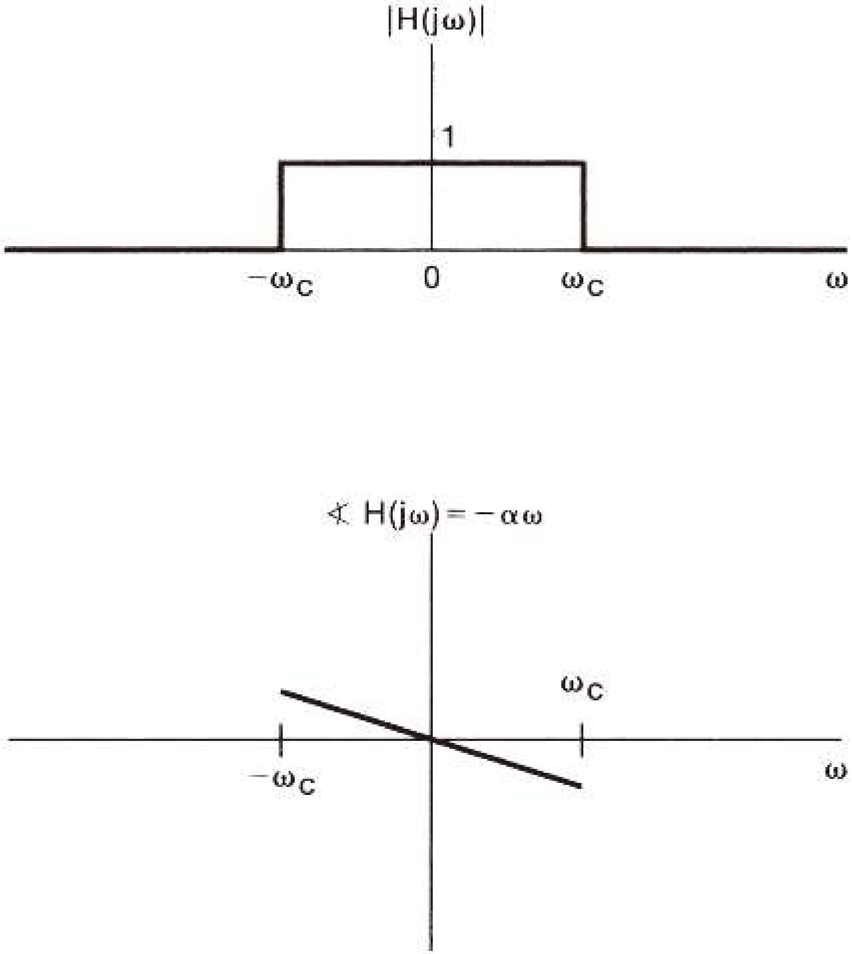


Figure 6.10 (a) The frequency response of a continuous-time ideal lowpass filter; (b) the frequency response of a discrete-time ideal lowpass filter.

Note zero phase

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

An ideal filter with linear phase over the passband, as illustrated in Figure 6.11, introduces only a simple **time shift** relative to the response of the ideal lowpass filter with zero phase characteristic.



6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

In particular, the impulse response corresponding to the filter in eq. (6.17) is

$$h(t) = \frac{\sin \omega_c t}{\pi t}, \tag{6.19}$$

連續時間理想低通濾波器的脈衝響應

Similarly, the impulse response of the discrete-time ideal filter in eq. (6.18) is

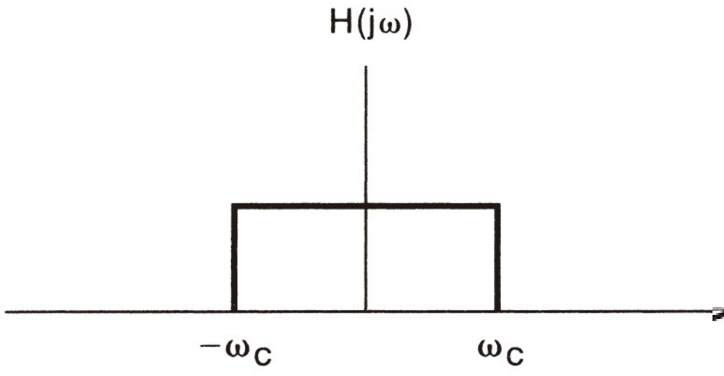
$$h[n] = \frac{\sin \omega_c n}{\pi n}, \tag{6.20}$$

離散時間理想低通濾波器的脈衝響應

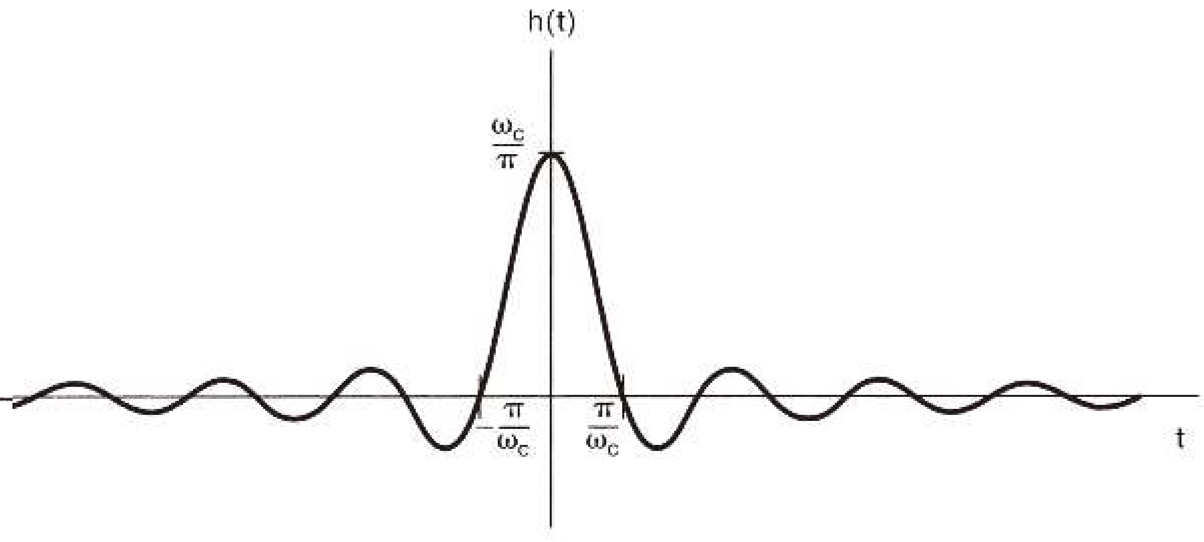
6.3 Time-Domain Properties of Ideal Frequency-Selective Filters

Inverse relation:

Note that in both contiunous and discrete time, the width of the filter passband is proportional to ω_c ,while the width of the main lobe of the impulse is proportional to $1/\omega_c$.

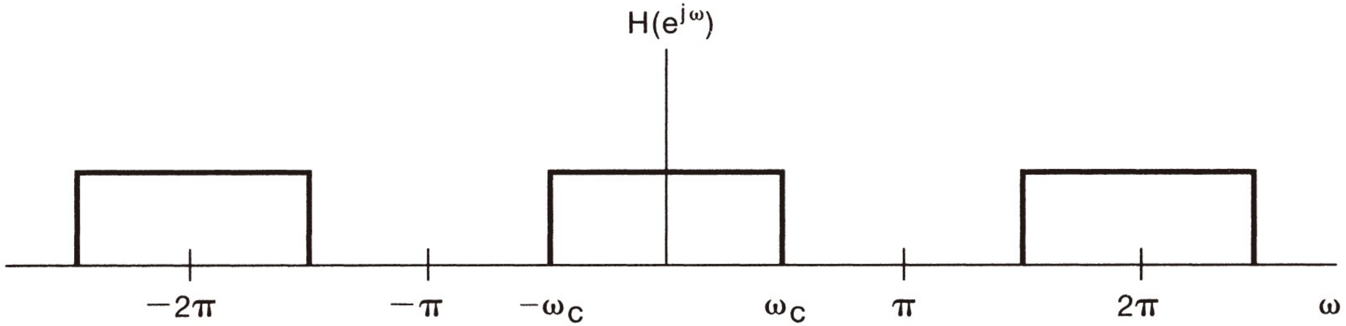


(a)



(a)

6.3 Time-Domain Properties of Ideal Frequency-Selective Filters



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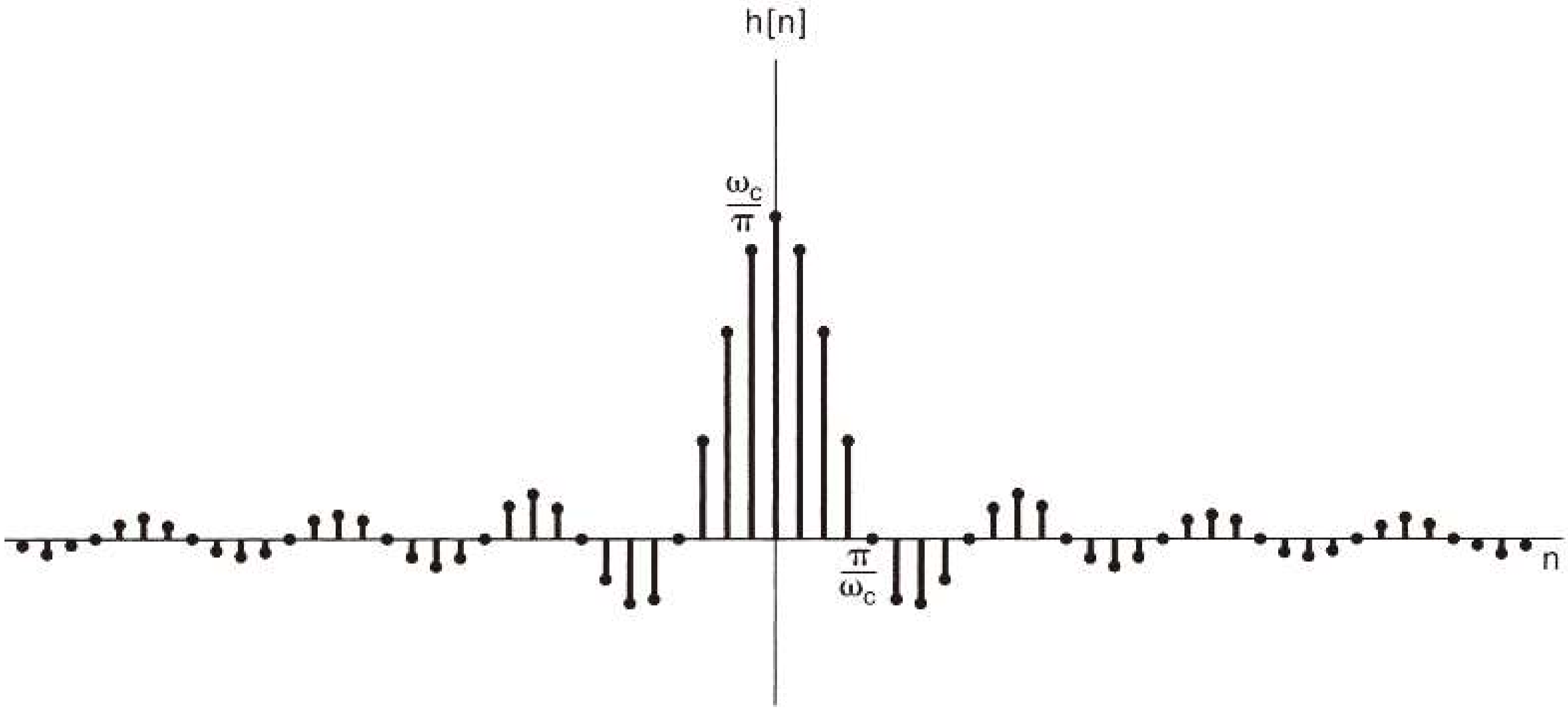

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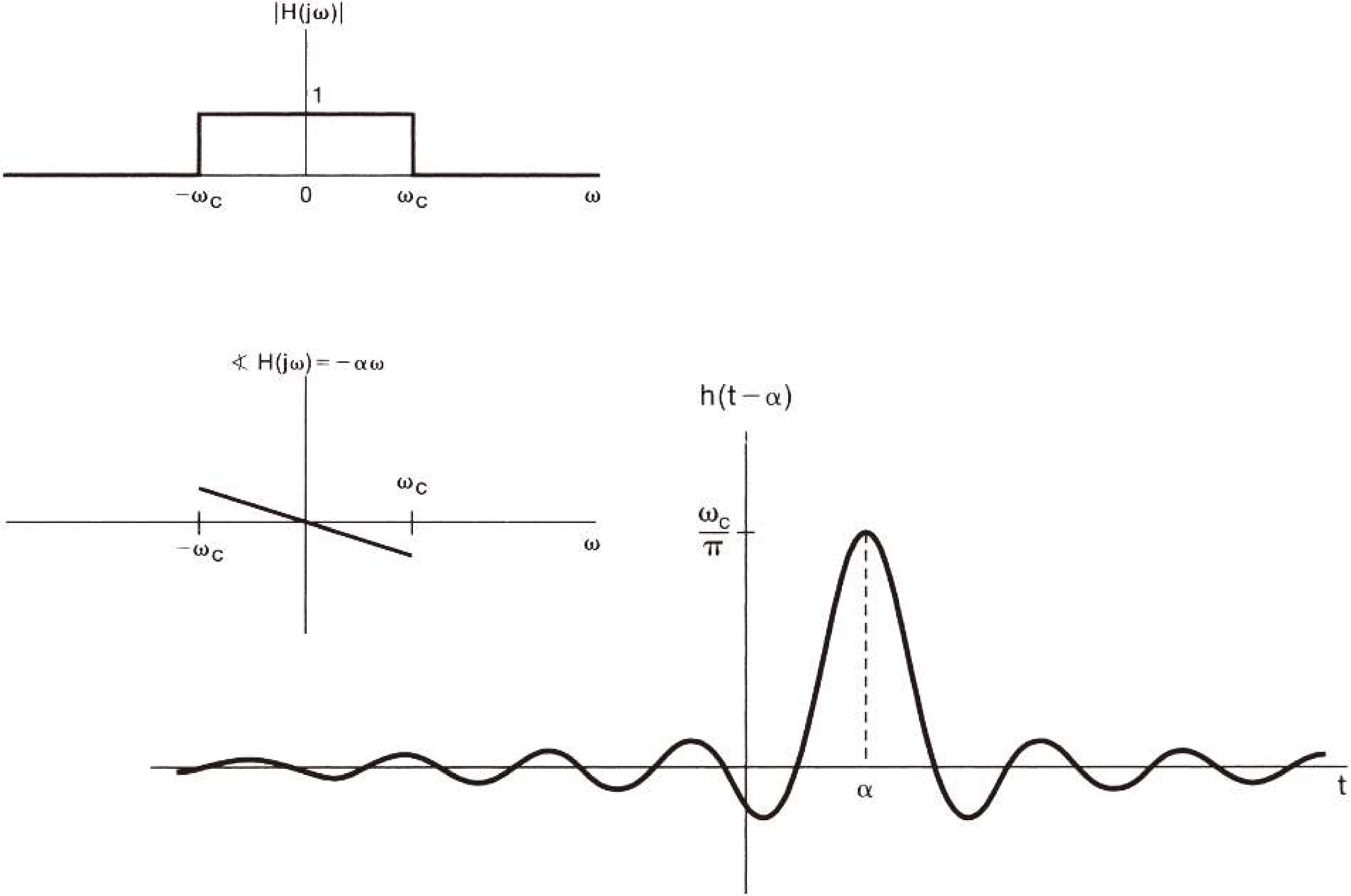
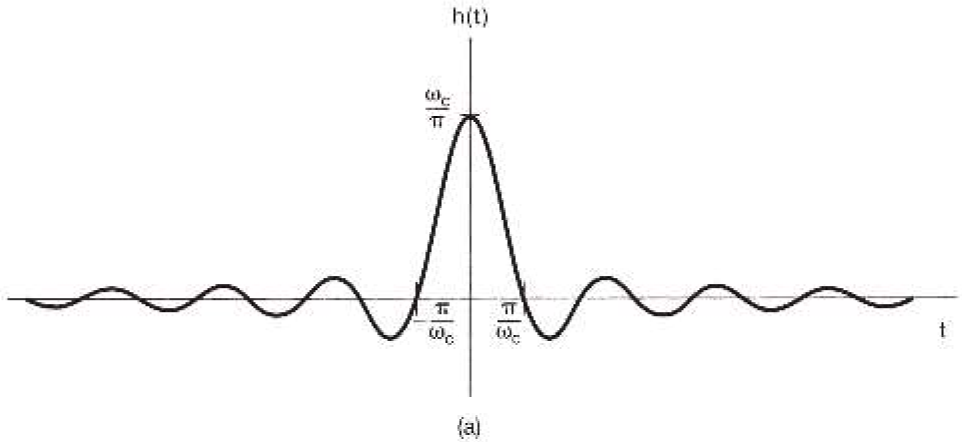
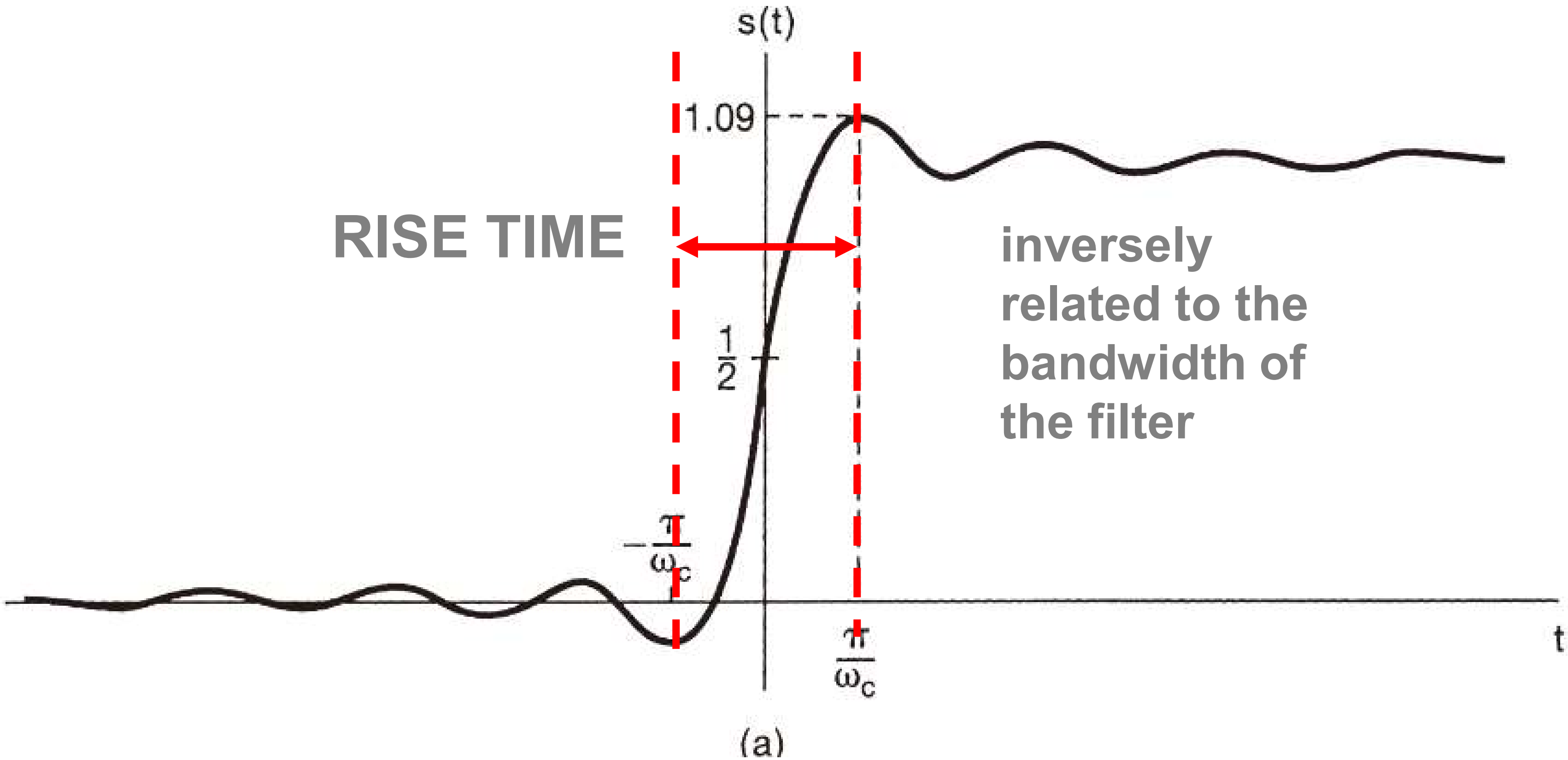


Figure 6.13 Impulse response of an ideal lowpass filter with magnitude and phase shown in Figure 6.11.

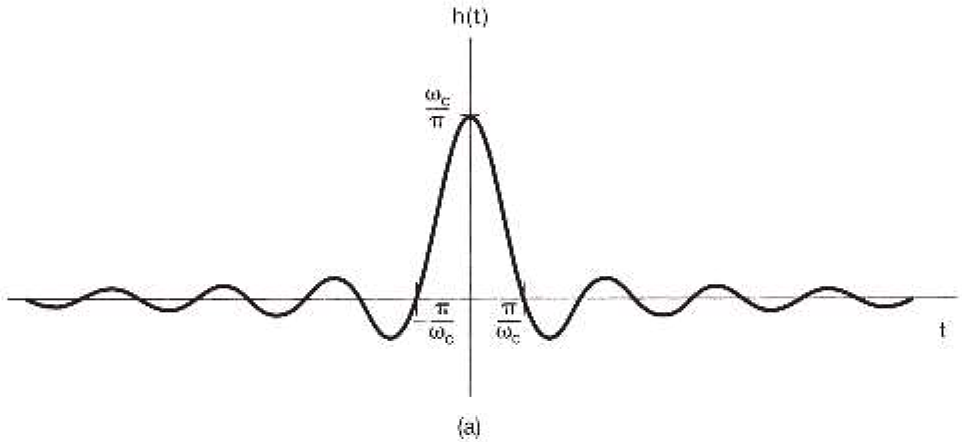
$$s(t)=\int_{t=-\infty}^{t} h(\tau)\,d\tau$$



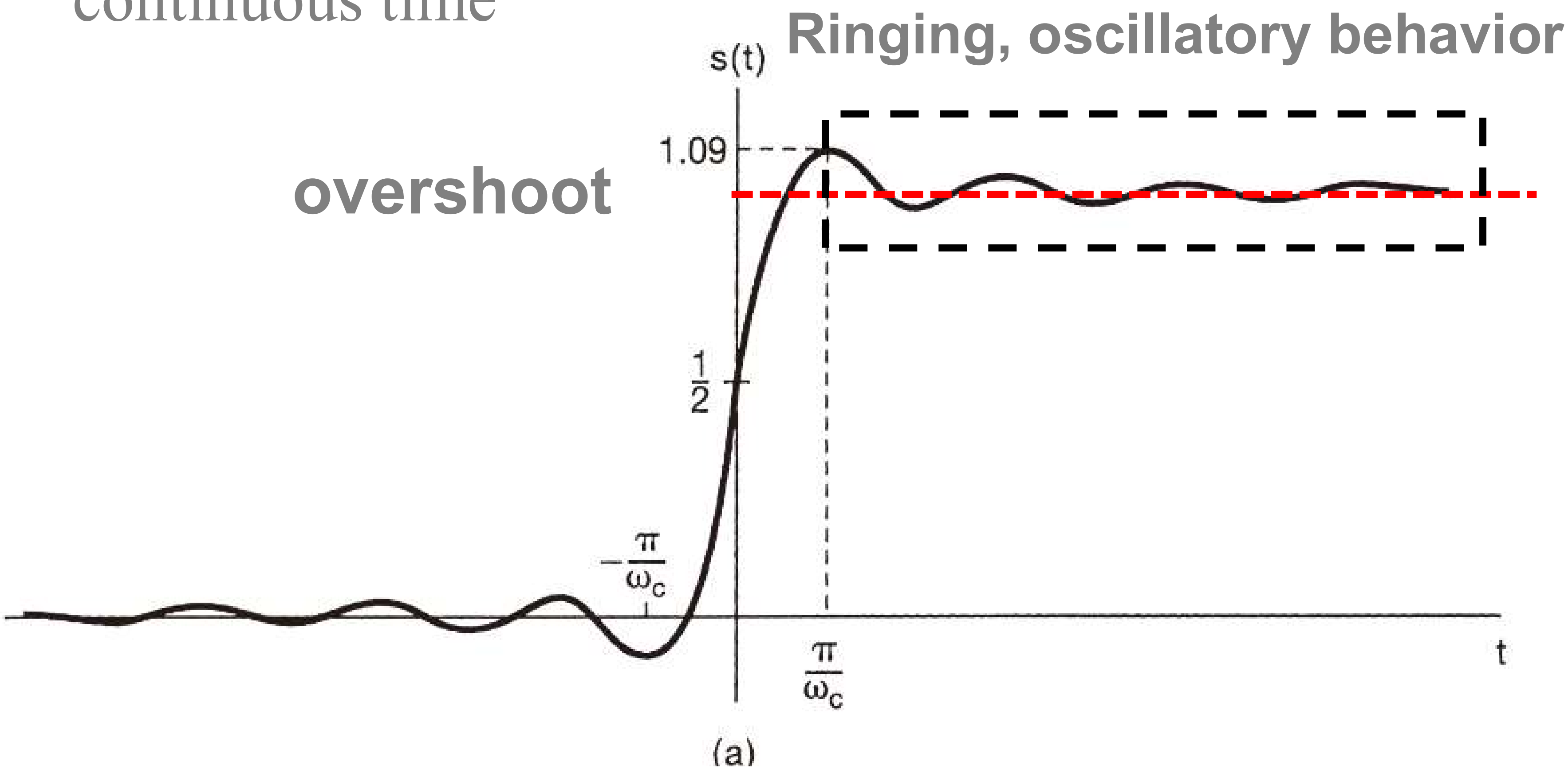
The step responses *s(t)* of the ideal lowpass filters in continuous time



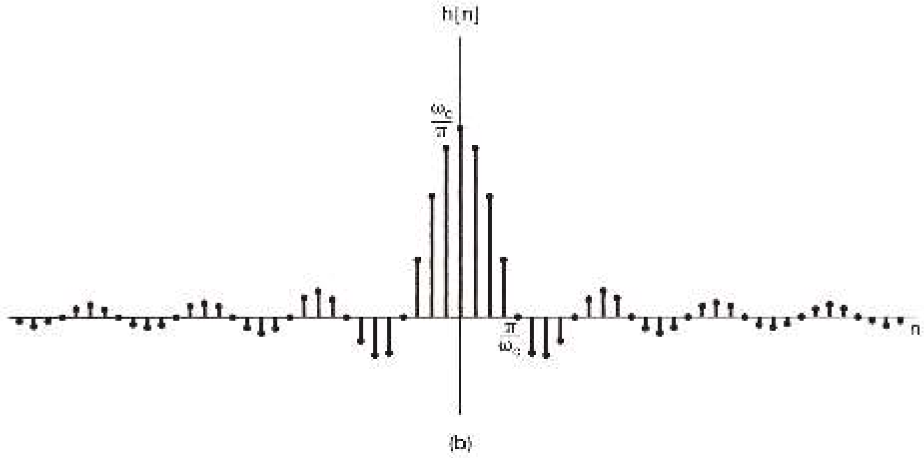
$$s(t) = \int_{t=-\infty}^{t} h(\tau)\, d\tau$$



The step responses *s(t)* of the ideal lowpass filters in continuous time



$$s[n] = \sum_{m=-\infty}^{n} h[m]$$



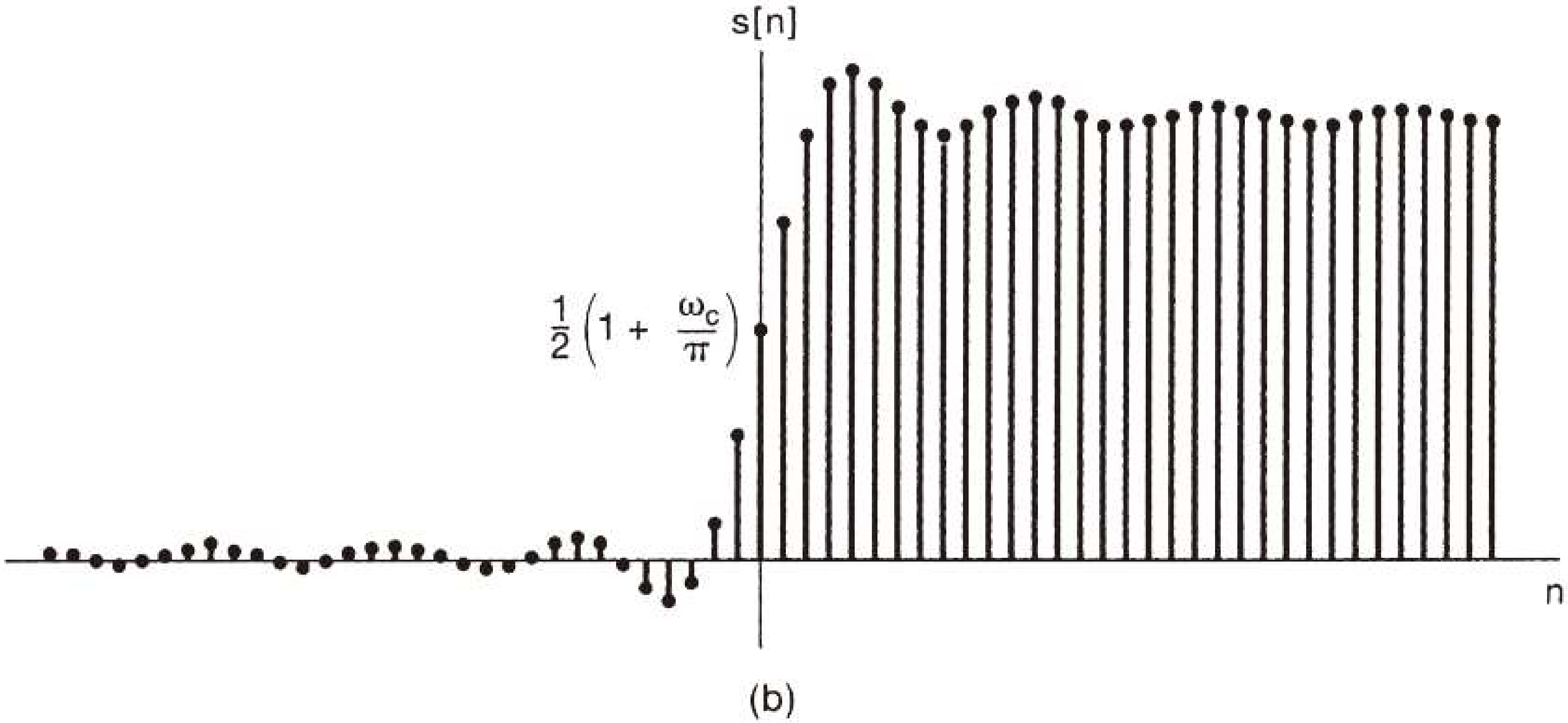


Figure 6.14 (a) Step response of a continuous-time ideal lowpass filter; (b) step response of a discrete-time ideal lowpass filter.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

The characteristics of ideal filter are not always desirable in practice. For example, in case where the spectra of two signal overlap slightly. A filter with gradual transition from passband to stopband is preferable.

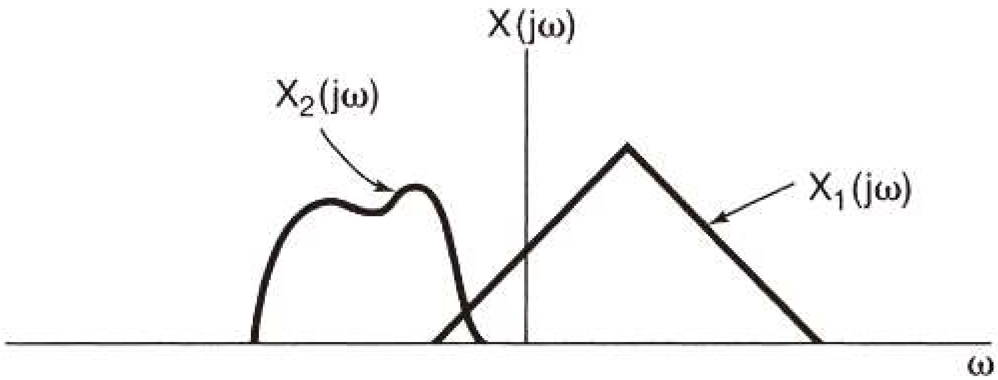
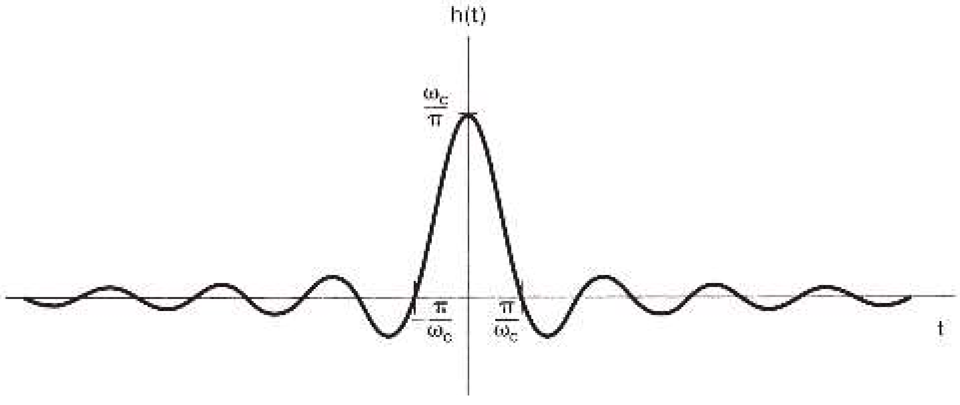


Figure 6.15 Two spectra that are slightly overlapping.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

Moreover, even in cases where the ideal frequency-selective characteristics are desirable, they may not be attainable. For example, idea filter is noncausal which is not attainable for real-time system.



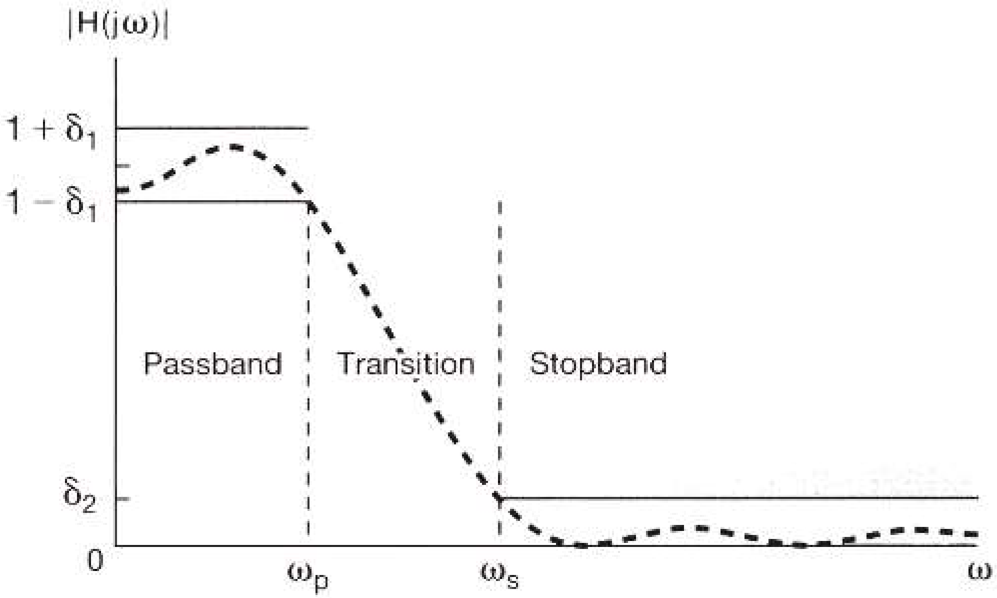
(a)

Moreover, it costs more to implement a more precise filter.

6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

For all of these reasons, nonideal filters are of considerable practical importance, and the characteristics of such filters are frequently specified or quantified in terms of several parameters in both the frequency and time domain.

ω_p: passband edge
ω_s: stopband edge

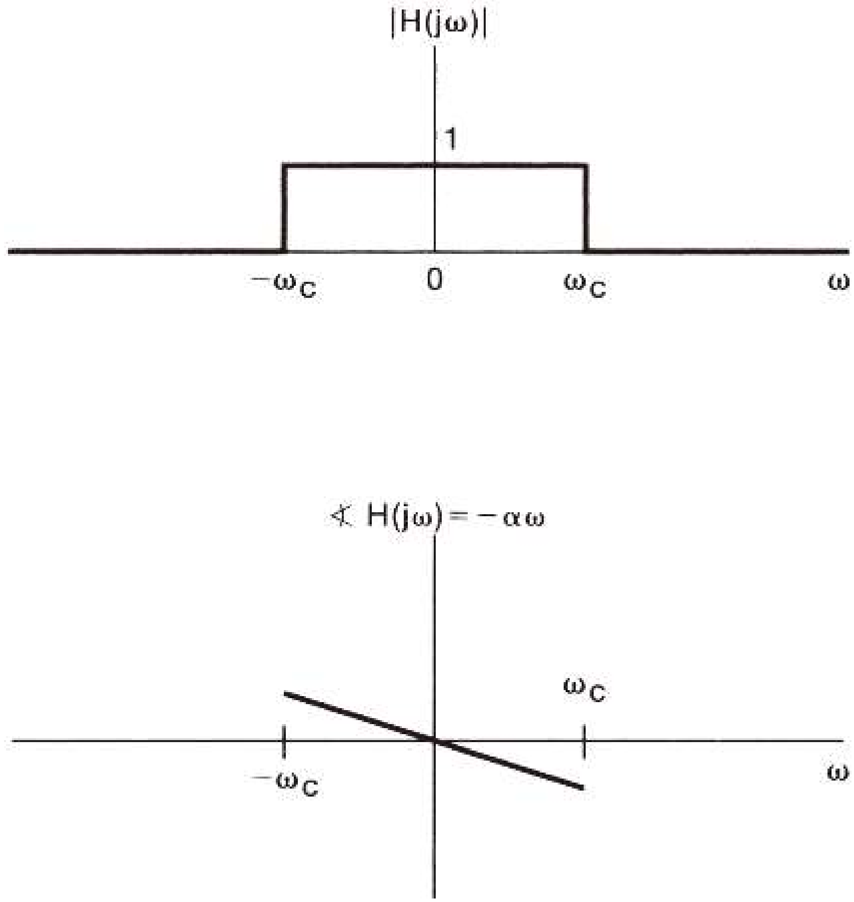


由圖 6.16 可知，通帶和止帶之間應存在逐漸過渡的地帶，且通帶和止帶在大小上亦應保留允許「通帶漣波」及「止帶漣波」的存在空間。

Figure 6.16 Tolerances for the magnitude characteristic of a lowpass filter. The allowable passband ripple is δ_1 and stopband ripple is δ_2. The dashed curve illustrates one possible frequency response that stays within the tolerable limits.

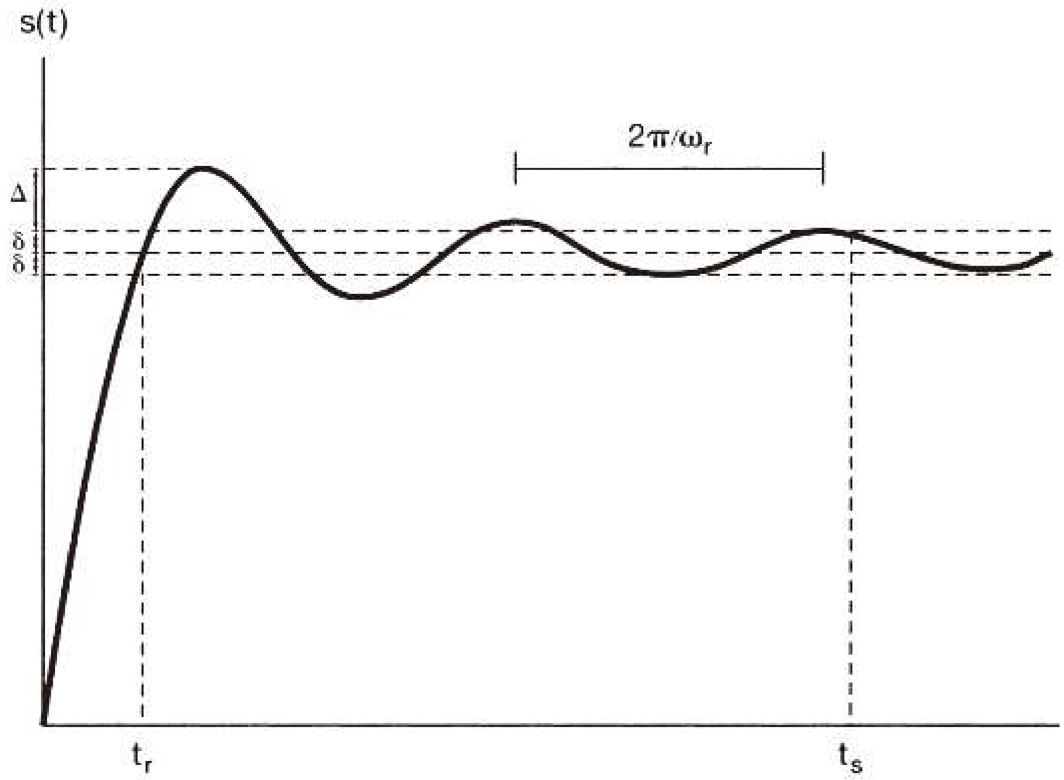
6.4 Time-Domain and Frequency-Domain Aspects of Nonideal Filters

In addition to the specification of magnitude characteristics in the frequency domain, in some cases the specification of phase characteristics is also important. In particular, a linear or nearly linear phase characteristic over the passband of the filter is desirable.



6.4 Time-Domain Aspects of Nonideal Filters

Rise time t_r of the step function –i.e., the interval over which the step response rises toward its final value. If ringing is present, then there are three other quantities that are often used to characterize the nature of these oscillations:



the overshoot △ of the final value of the step response, the ringing frequency ω_r ,and the settling time t_s—i.e., the time required for the step response to settle to within a specified tolerance δ of its final value.

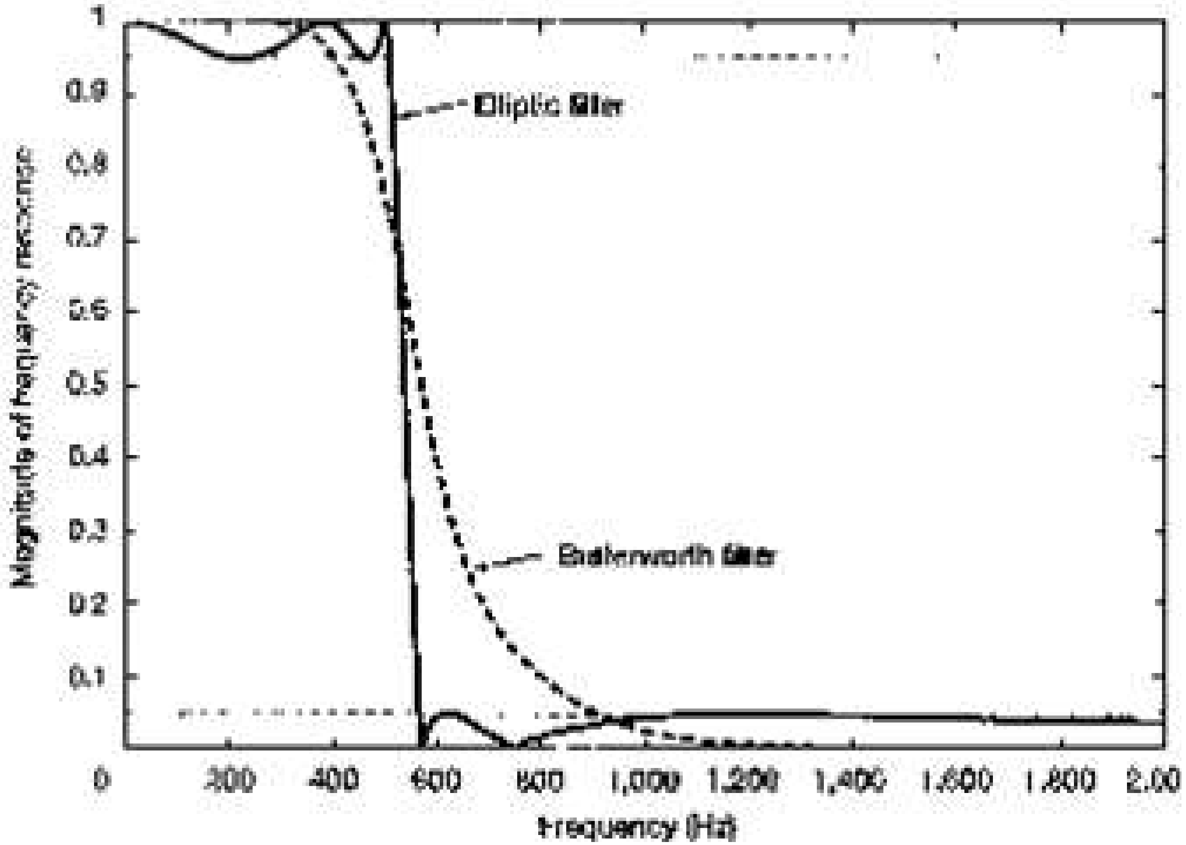
Trade-off & Example 6.3

There may be a trade-off between the width of the transition band and the settling time of the step response.

Let us consider two specific lowpass filter designed to have a **cutoff** frequency of 500 Hz. Each filter has a fifth-order rational frequency response and a real-valued impulse response. The two filters are of specific types, one referred to as **Butterworth** filters and the other as **elliptic** filters. Both of these classes of filters are frequently used in practice.

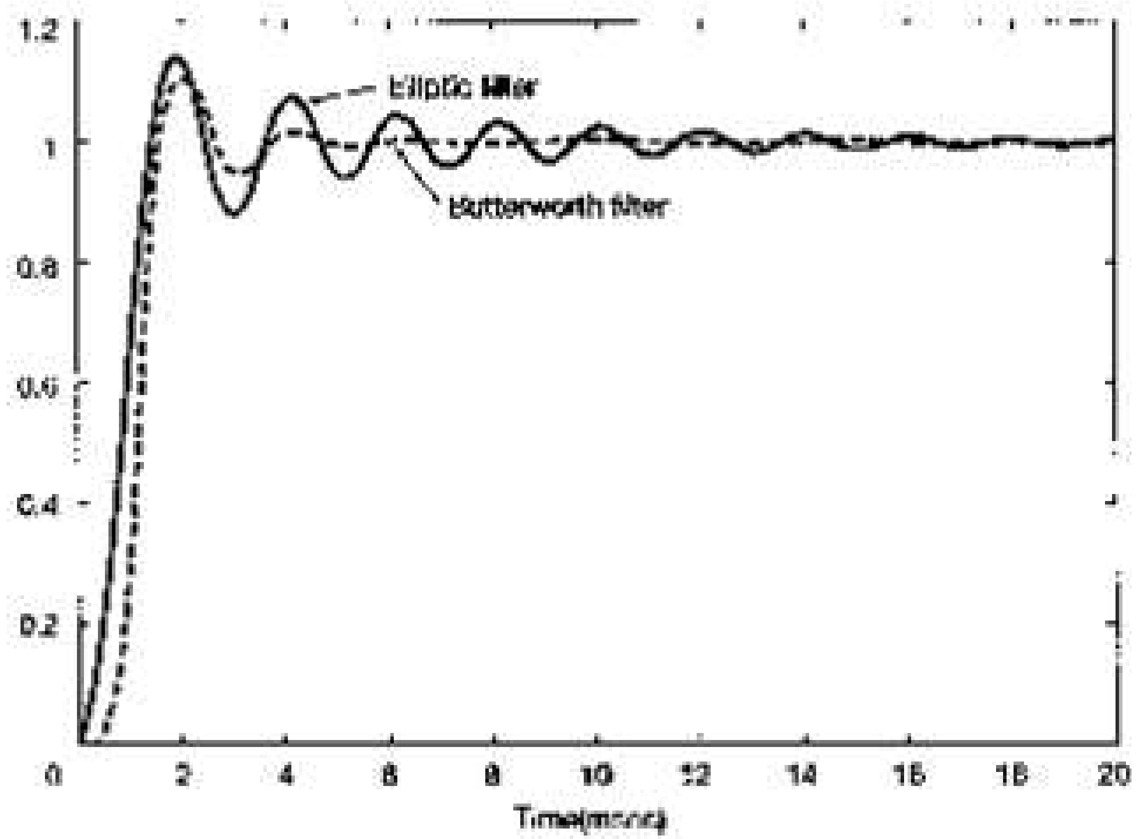
Example 6.3

The magnitudes of the frequency responses of the two filters are plotted (versus frequency measured in Hertz) in Figure 6.18(a). We take the transition band of each filter as the region around the cutoff frequency (500 Hz) where the frequency response magnitude is neither within .05 of unity magnitude



(the passband ripple) nor within .05 of zero magnitude (the stopband ripple). From Figure 6.18(a), it can be seen that the transition band of the Butterworth filter is wider than the transition band of the elliptic filter.

Example 6.3



The price paid for the narrower transition band of the elliptic filter may be observed in Figure 6.18(b), in which the step responses of both filters are displayed.

We see that the ringing in the elliptic filter's step response is more prominent than for the Butterworth step response. In particular, the settling time for the step response is longer in the case of the elliptic filter.

6.5 First-Order and Second-Order Continuous-Time Systems

Higher-order system is typically implemented by combining first-order and second-order systems in cascade or parallel arrangements. Hence, we need to understand the time and frequency domain properties of them.

6.5.1 First-Order Continuous-Time Systems

The differential equation for a first-order system is often expressed in the form

(6.21)

$$\tau \frac{dy(t)}{dt} + y(t) = x(t),$$

The corresponding frequency response for the first-order system is

(6.22)

$$H(j\omega) = \frac{1}{j\omega\tau + 1},$$

τ is the time constant of the system

6.5.1 First-Order Continuous-Time Systems

$$H(j\omega)=\frac{1}{j\omega\tau+1},$$

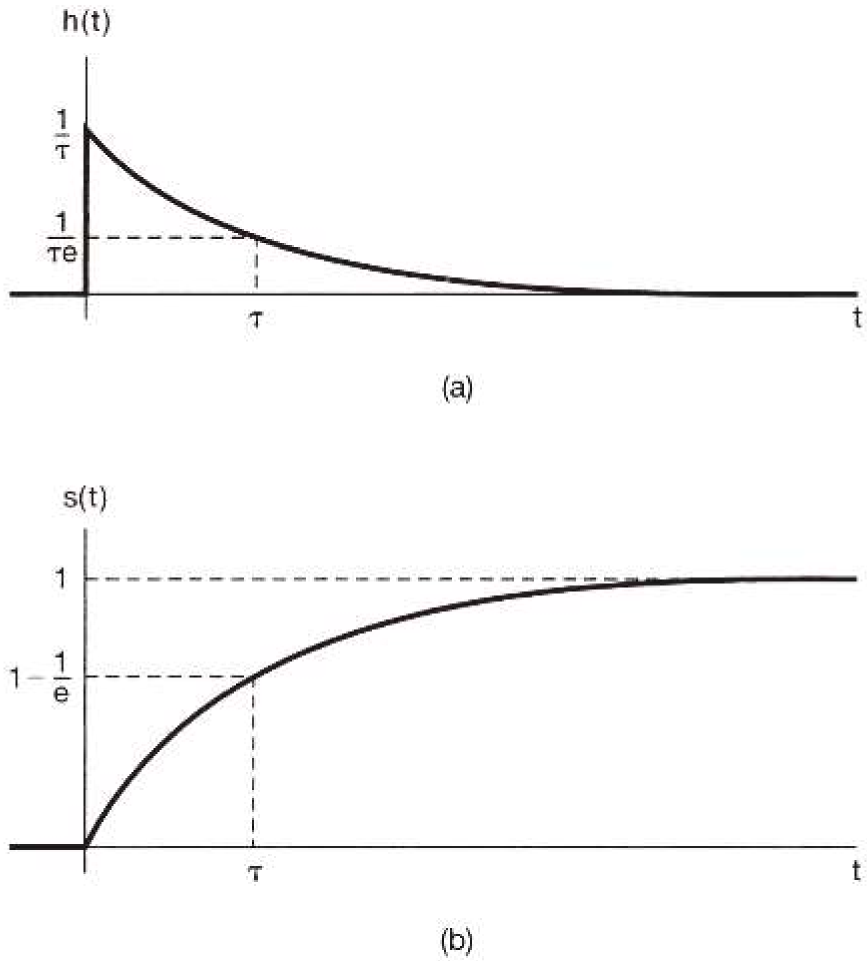
and the impulse response is

(6.23)

$$h(t)=\frac{1}{\tau}e^{-t/\tau}u(t),$$

The step response of the system is

$$s(t)=h(t)*u(t)$$
$$=\left[1-e^{-t/\tau}\right]u(t).$$ (6.24)



τ is the time constant of the system

6.5.1 First-Order Continuous-Time Systems

$$H(j\omega)=\frac{1}{j\omega\tau+1},$$

Specifically, from eq. (6.22), we obtain

$$20\log_{10}|H(j\omega)|=-10\log_{10}\left[(\omega\tau)^2+1\right] \qquad (6.25)$$

From this we see that for $\omega\tau<<1$,the log magnitude is approximately zero, while for $\omega\tau>>1$,the log magnitude is approximately a linear function of $\log_{10}(\omega)$. That is,

$$20\log_{10}|H(j\omega)|\approx 0 \qquad for \qquad \omega<<1/\tau, \qquad (6.26)$$

and

$$20\log_{10}|H(j\omega)|\approx -20\log_{10}(\omega\tau)$$

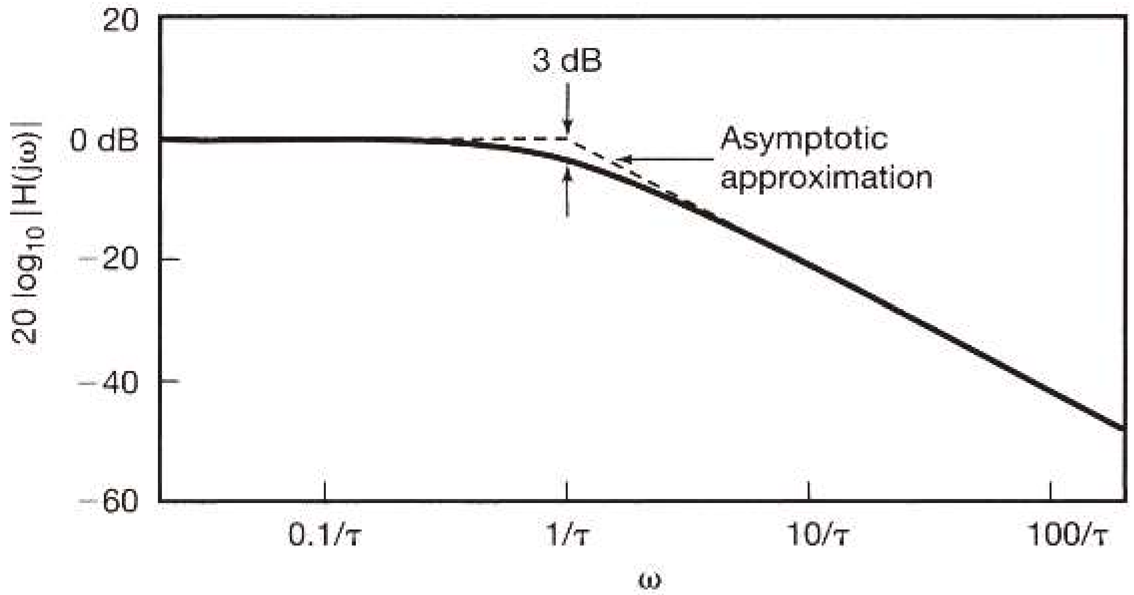
$$=-20\log_{10}(\omega)-20\log_{10}(\tau) \qquad for \qquad \omega>>1/\tau.$$

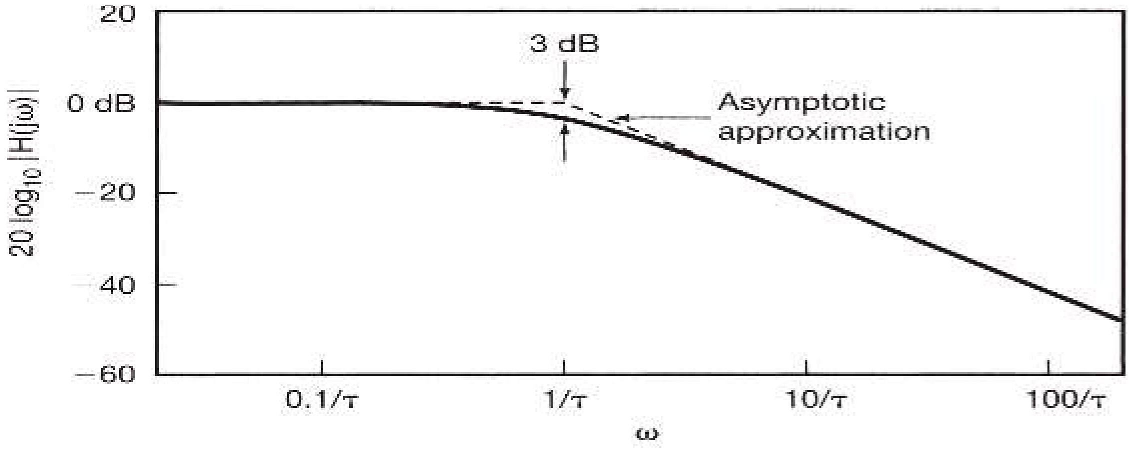
6.5.1 First-Order Continuous-Time Systems

$$20\log_{10}|H(j\omega)| \approx 0 \quad for \quad \omega << 1/\tau,$$

$$20\log_{10}|H(j\omega)| \approx -20\log_{10}(\omega) - 20\log_{10}(\tau) \quad for \quad \omega >> 1/\tau.$$

For the first-order system, the low- and high-frequency asymptotes of the log magnitude are straight lines, where the magnitude decrease 20dB for every 10 times ω





The point at which the slope of the approximation changes is precisely $\omega = 1/\tau$.

$\omega = 1/\tau$ referred to as break frequency

note that at $\omega = 1/\tau$ the two terms [$(\omega\tau)^2$ and 1] in the argument of the logarithm in eq. (6.25) are equal.

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left[(\omega\tau)^2 + 1\right].$$

$$20\log_{10}\left|H\left(j\frac{1}{\tau}\right)\right| = -10\log_{10}(2) \approx -3dB. \quad (6.28)$$

referred to as 3-dB point.

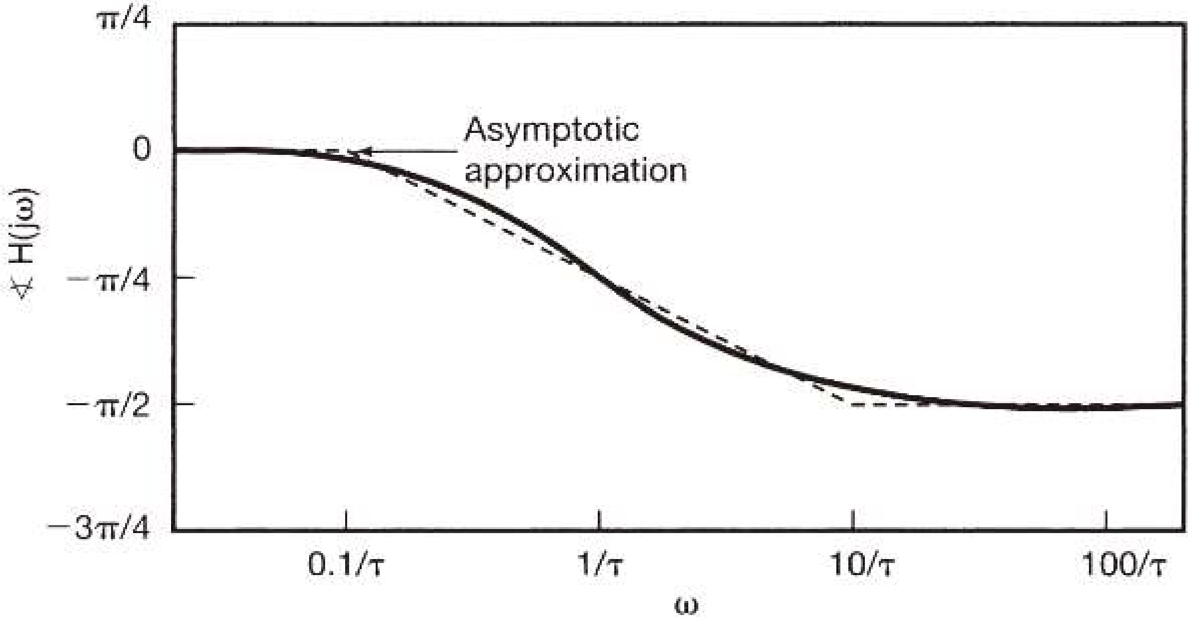
6.5.1 First-Order Continuous-Time Systems

$$H(j\omega)=\frac{1}{j\omega\tau+1},$$

It is also possible to obtain a useful straight-line approximation to

$$\sphericalangle H(j\omega)=-\tan^{-1}(\omega\tau) \qquad (6.29)$$

$$\approx\begin{cases}0, & \omega\le 0.1/\tau \\ -(\pi/4)[\log_{10}(\omega\tau)+1], & 0.1/\tau\le\omega\le 10/\tau. \\ -\pi/2, & \omega\ge 10/\tau\end{cases}$$



6.5.1 First-Order Continuous-Time Systems

$$H(j\omega) = \frac{1}{j\omega\tau + 1},$$

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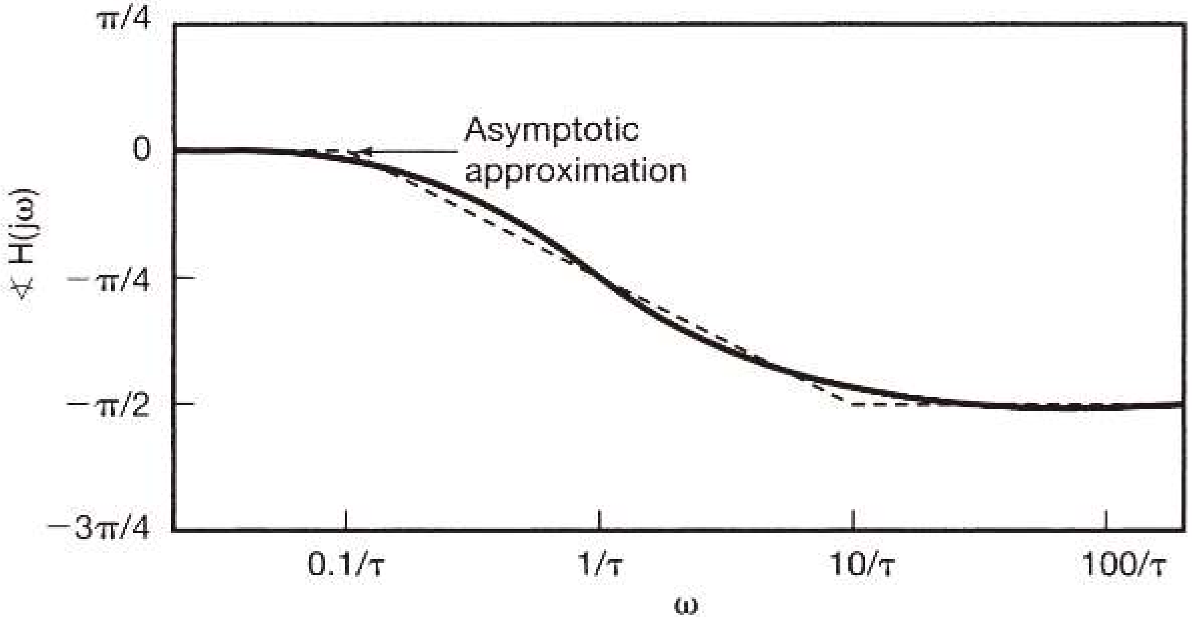


6.5.1 First-Order Continuous-Time Systems

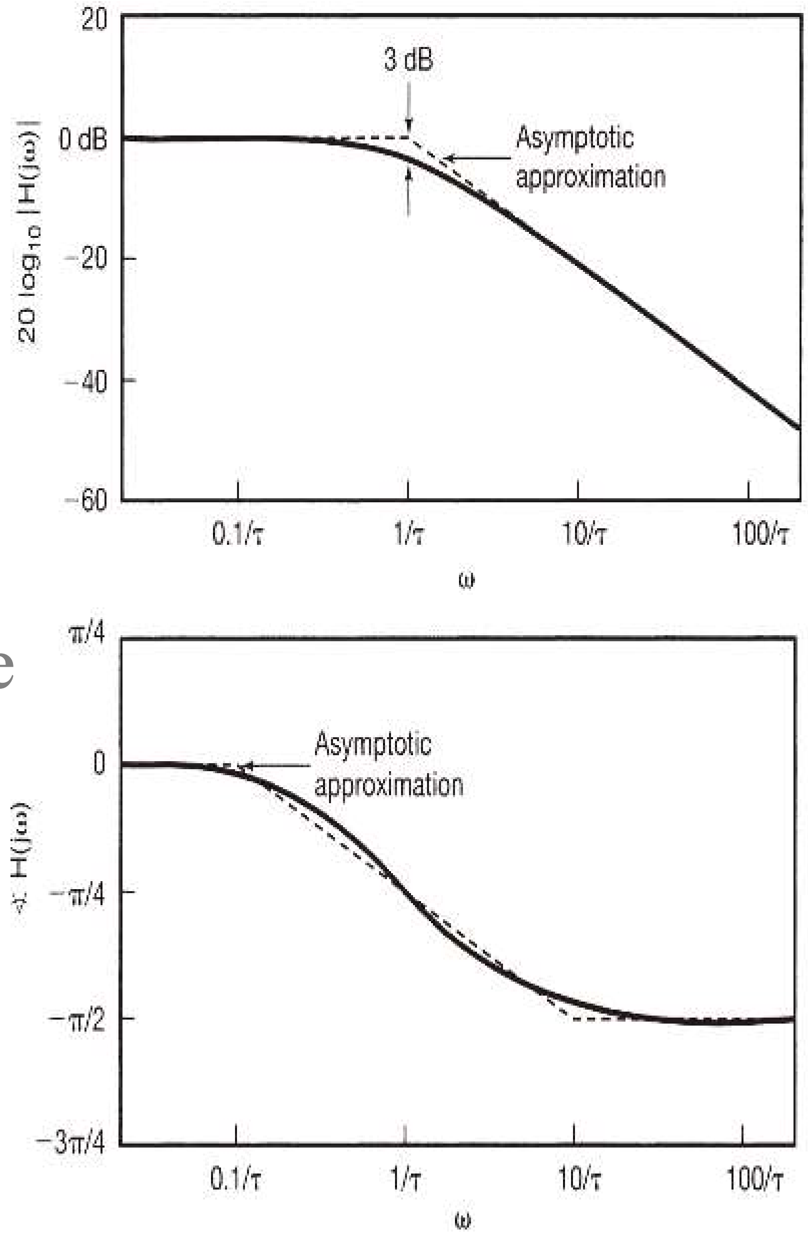
Note that this approximation decreases linearly (from 0 to $-\pi/2$ as a function of $\log_{10}(\omega)$ in the range

$$\frac{0.1}{\tau} \le \omega \le \frac{10}{\tau},$$

the approximation agrees with the actual value of $\measuredangle H(j\omega)$ at the break frequency $\omega = 1/\tau$, at which point

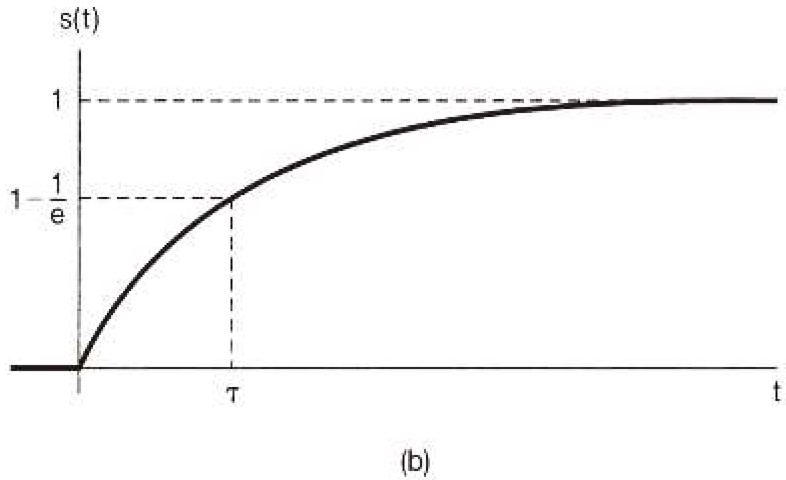
$$\measuredangle H(j\omega) = -\tan^{-1}(\omega\tau) \quad (6.30)$$

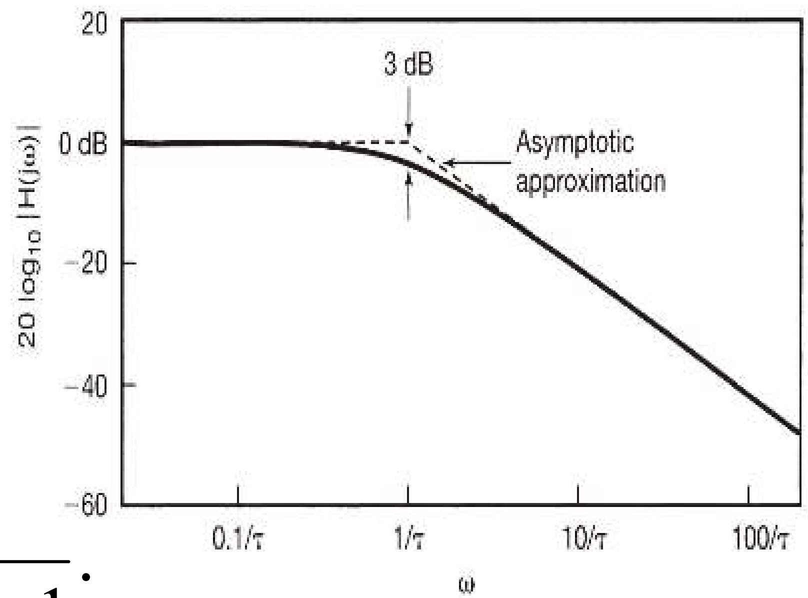
$$\measuredangle H\left(j\frac{1}{\tau}\right) = -\frac{\pi}{4}.$$



6.5.1 First-Order Continuous-Time Systems

As we make τ smaller, we speed up the time response of the system [i.e., *h(t)* becomes more compressed toward the origin, and the rise time of the step response is reduced] and we simultaneously make the break frequency large [i.e., *H(jω)* becomes broader, since $|H(j\omega)| \approx 1$ for a larger range of frequencies].





$$\tau h(t) = e^{-t/\tau}u(t), \qquad H(j\omega) = \frac{1}{j\omega\tau + 1}.$$

6.5.2 Second-Order Continuous-Time Systems

The linear constant-coefficient differential equation for a **second-order** system is

$$\frac{d^2 y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t) \quad (6.31)$$

this type of equations arise in many physical system such as

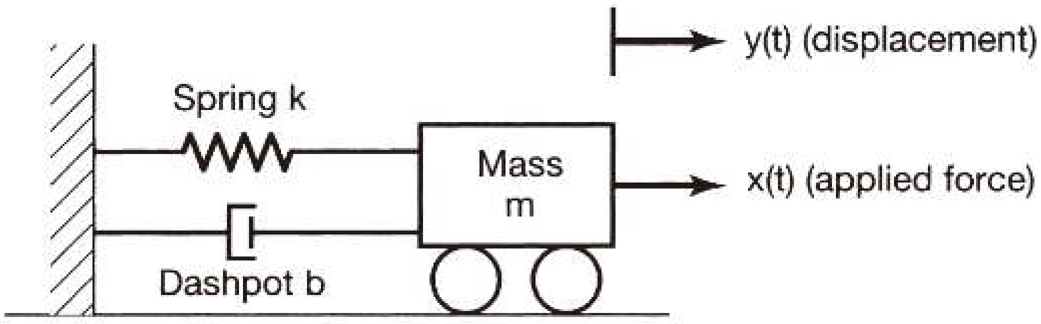


Figure 6.21 Second-order system consisting of a spring and dashpot attached to a moveable mass and a fixed support.

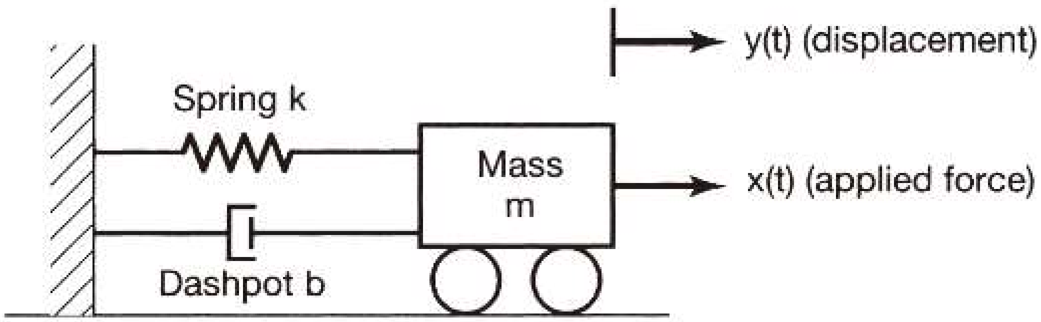


Figure 6.21 Second-order system consisting of a spring and dashpot attached to a moveable mass and a fixed support.

In the figure, the input is the applied force *x(t)* and the output is the displacement of the mass *y(t)* from some equilibrium position at which the spring exerts no restoring force.

$$m\frac{d^2y(t)}{dt^2} = x(t) - ky(t) - b\frac{dy(t)}{dt},$$

$$\omega_n = \sqrt{\frac{k}{m}}$$

or $$\frac{d^2y(t)}{dt^2} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \frac{1}{m}x(t).$$

$$\zeta = \frac{b}{2\sqrt{km}},$$

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n^2 y(t) = \frac{\omega_n^2}{k}x(t).$$

$$\frac{d^2y(t)}{dt^2} + 2\zeta\omega_n \frac{dy(t)}{dt} + \omega_n^2 y(t) = \omega_n^2 x(t)$$

The frequency response for the second-order system of eq. (6.31) is

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}. \qquad (6.33)$$

The denominator of *H(jω)* can be factored to yield

$$\mathrm{H}(j\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)},$$

where

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}. \qquad (6.34)$$

$$h(j\omega) = \frac{\omega_n^2}{(j\omega - c_1)(j\omega - c_2)}, \qquad \begin{aligned} c_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \\ c_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}. \end{aligned}$$

For $\zeta \neq 1$, c_1 and c_2 are unequal, and we can perform a partial-fraction expansion of the form

$$H(j\omega) = \frac{M}{j\omega - c_1} - \frac{M}{j\omega - c_2}, \tag{6.35}$$

where

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}. \tag{6.36}$$

$$e^{-at}u(t), \Re e\{a\} > 0 \qquad \frac{1}{a + j\omega}$$

From eq. (6.35), the corresponding impulse response for the system is

$$h(t) = M\left[e^{c_1 t} - e^{c_2 t}\right]u(t). \tag{6.37}$$

6.5.2 Second-Order Continuous-Time Systems

If $\zeta = 1$, then $c_1 = c_2 = -\omega_n$, and

$$H(j\omega) = \frac{\omega_n^2}{(j\omega + \omega_n)^2}. \tag{6.38}$$

From Table 4.2,

$te^{-at}u(t), \Re e\{a\} > 0$	$\dfrac{1}{(a + j\omega)^2}$

we find that in this case the impulse response is

$$h(t) = \omega_n^2 te^{-\omega_n t}u(t). \tag{6.39}$$

$$h(t) = \omega_n^2 t e^{-\omega_n t} u(t).(6.39)$$

$$\frac{h(t)}{\omega_n} = \omega_n t e^{-\omega_n t} u(t)$$

Note from eqs. (6.37) and (6.39), that h(t)/ ω_n is a function of $\omega_n t$.

$$h(t) = M\left\lfloor e^{c_1 t} - e^{c_2 t} \right\rfloor u(t).(6.37) \quad c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$

$$\frac{h(t)}{\omega_n} = \frac{1}{2\sqrt{\zeta^2 - 1}}\left[e^{c_1 t} - e^{c_2 t}\right]u(t). \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}.$$

ζ **Damping ratio**

ω_n **Undamped natural frequency**

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}.$$

$$\frac{h(t)}{\omega_n} = \frac{1}{2\sqrt{\zeta^2 - 1}}\left[e^{c_1 t} - e^{c_2 t}\right]u(t). \qquad \begin{aligned} c_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \\ c_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}. \end{aligned}$$

we see that for $0 < \zeta < 1$,

c_1 and c_2 are complex, and we can rewrite the impulse response in eq. (6.37) in the form

$$\begin{aligned} c_1 &= -\zeta\omega_n + \omega_n j\sqrt{1-\zeta^2}, \\ c_2 &= -\zeta\omega_n - \omega_n j\sqrt{1-\zeta^2}. \end{aligned} \qquad (6.40)$$

$$\frac{h(t)}{\omega_n} = \frac{e^{-\zeta\omega_n t}}{2j\sqrt{1-\zeta^2}}\left\{\exp\left[j(\omega_n\sqrt{1-\zeta^2})t\right] - \exp\left[-j(\omega_n\sqrt{1-\zeta^2})t\right]\right\}u(t)$$

$$= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_n\sqrt{1-\zeta^2})t\right]u(t).$$

under-damped

$$\frac{h(t)}{\omega_n} = \frac{1}{2\sqrt{\zeta^2 - 1}}\left[e^{c_1 t} - e^{c_2 t}\right]u(t).$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}.$$

we see that for $\zeta > 1$,

c_1 and c_2 are real and negative, and we can rewrite the impulse response in the form

$$\frac{h(t)}{\omega_n} = \frac{1}{2\sqrt{\zeta^2 - 1}}\left\{\exp\left[(-\zeta + \sqrt{\zeta^2 - 1})\omega_n t\right] - \exp\left[(-\zeta - \sqrt{\zeta^2 - 1})\omega_n t\right]\right\}u(t)$$

which is the difference between two decaying exponentials.

over-damped

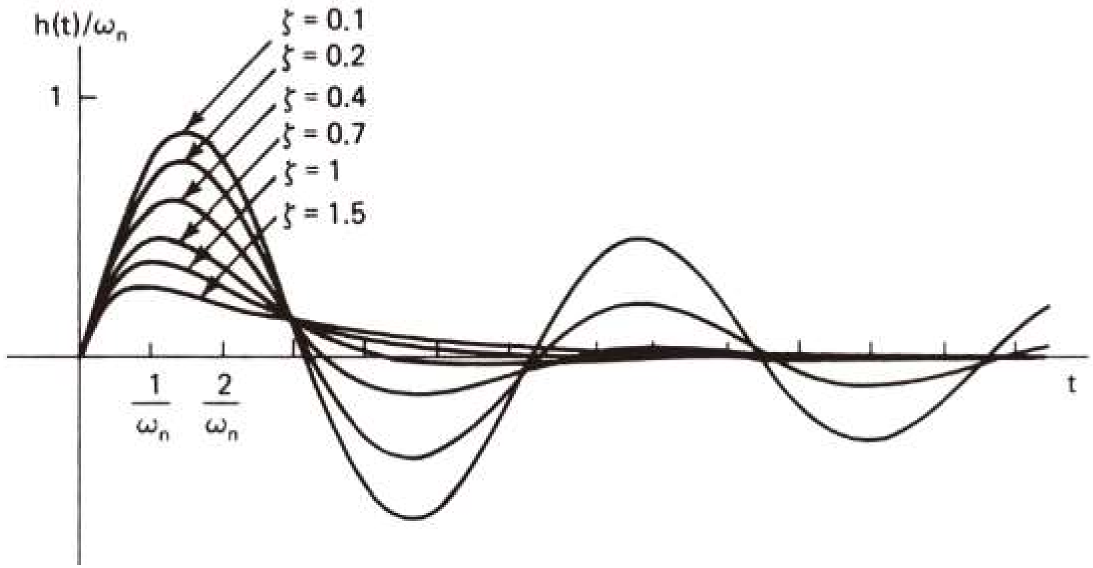
The case of $\zeta = 1$, when $c_1 = c_2$, is called the **critically damped** case.

$$\frac{h(t)}{\omega_n} = \omega_n t e^{-\omega_n t} u(t) \qquad \frac{h(t)}{\omega_n} = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left[\sin(\omega_n\sqrt{1-\zeta^2})t\right]u(t).$$

under-damped

$$\frac{h(t)}{\omega_n} = \frac{1}{2\sqrt{\zeta^2-1}}\left\{\exp\left[(-\zeta+\sqrt{\zeta^2-1})\omega_n t\right]-\exp\left[(-\zeta-\sqrt{\zeta^2-1})\omega_n t\right]\right\}u(t)$$

over-damped



6.5.2 Second-Order Continuous-Time Systems

The step response of a second-order system can be calculated from eq. (6.37) for $\zeta \neq 1$. This yields the expression

$$s(t) = h(t) * u(t) = \int_{-\infty}^{t} h(\tau) = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\} u(t). \quad (6.41)$$

$$s(t) = \int_{-\infty}^{t} h(\tau) = u(t)\left(\int_{0}^{t} M\left[e^{c_1\tau} - e^{c_2\tau}\right]\right) = M\left[\frac{e^{c_1\tau}}{c_1} - \frac{e^{c_2\tau}}{c_2}\right]\Bigg|_0^t u(t)$$

$$= \left\{M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right] - M\left(\frac{1}{c_1} - \frac{1}{c_2}\right)\right\} u(t) = \left\{M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right] + 1\right\} u(t)$$

For $\zeta = 1$, we can use eq. (6.39) to obtain

$$s(t) = \left[1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}\right] u(t). \quad (6.42)$$

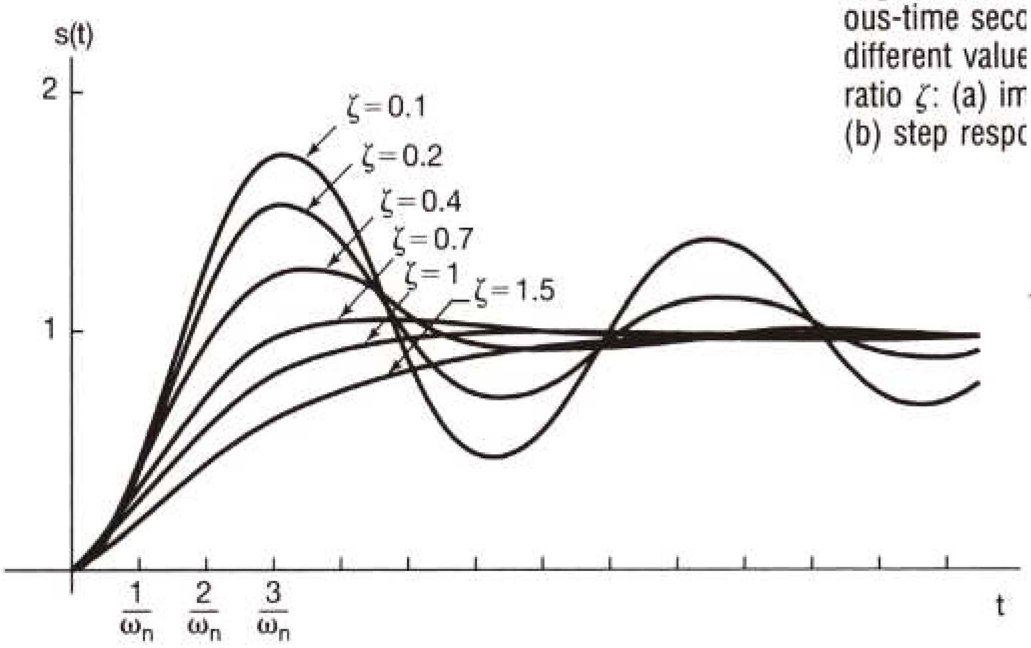


Figure 6.22
ous-time secc
different value
ratio ζ: (a) im
(b) step respc

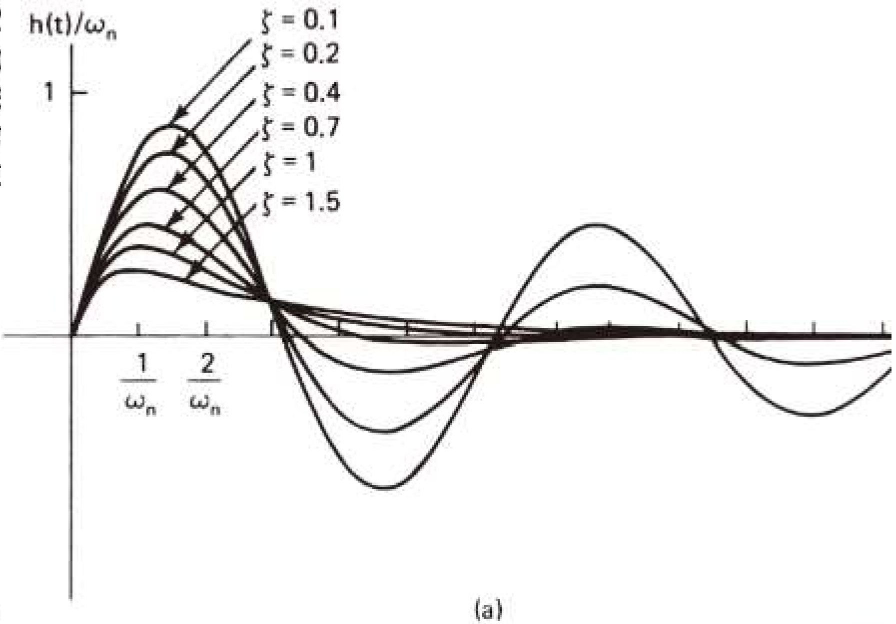


Fig
ous

Fot $0<\zeta<1$, overshoot and ringing.

For $\zeta = 1$, the fastest response (i.e., the shortest rise time) that is possible without overshoot and thus has the shortest settling time.

Fot $1<\zeta$, the bigger the ζ, the longer the rise/settling time

$$s(t) = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\} u(t).$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1},$$

$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}.$$

6.5.2 Second-Order Continuous-Time Systems

$$H(j\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}.(6.33)$$

Furthermore, eq. (6.33) can be rewritten as

$$H(j\omega)=\frac{1}{(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1},$$

Figure 6.23 depicted the Bode plot of the frequency response given in eq. (6.33) for several values of ζ .

$$20\log_{10}|H(j\omega)|=-10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}. \qquad (6.41)$$

From this expression, it follows that

$$20\log_{10}|H(j\omega)|\cong\begin{cases}0, & for\,\omega\langle\langle\omega_n \\ -40\log_{10}\omega+40\log_{10}\omega_n, & for\,\omega\rangle\rangle\omega_n\end{cases}. \qquad (6.42)$$

$$20\log_{10}|H(j\omega)| = -10\log_{10}\left\{\underline{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2}\right\}.$$

$$a = \left(\frac{w}{w_n}\right)^2 \qquad (1-a)^2 + 4\zeta^2 a = a^2 + (4\zeta^2 - 2)a + 1 = f(a)$$

$$a^* = \operatorname{argmax} f(a) \qquad f'(a) = 2a + 4\zeta^2 - 2 = 0$$

$$a^* = 1 - 2\zeta^2$$

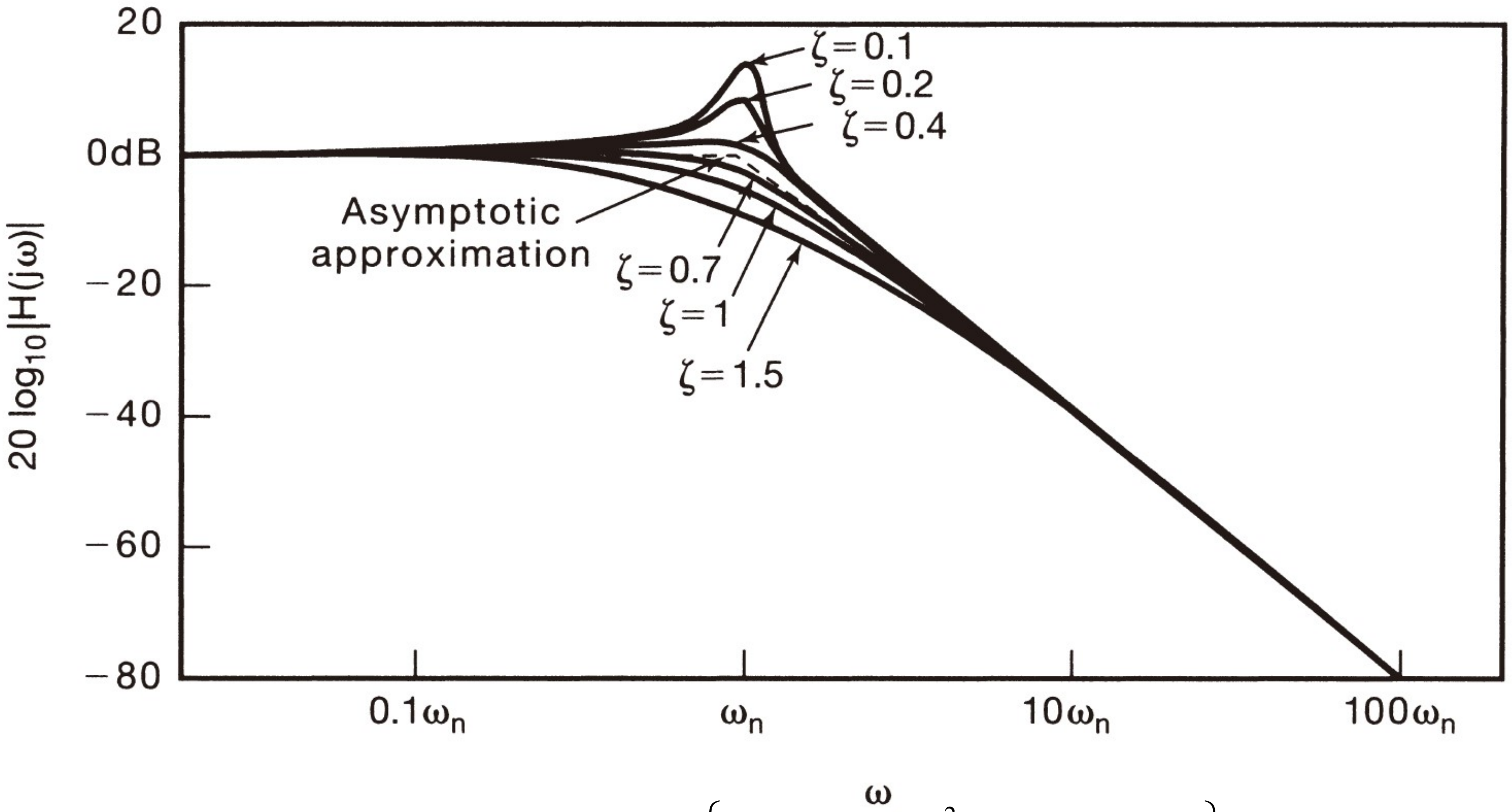
$$\left(\frac{w^*}{w_n}\right)^2 = 1 - 2\zeta^2 \Rightarrow w^* = w_n\sqrt{1-2\zeta^2}$$

IF $1 - 2\zeta^2 < 0$, $-1 + 2\zeta^2 > 0$

$$f'(a) = -2(1 - 2\zeta^2 - a) = 2(a + 2\zeta^2 - 1) > 0$$

6.5.2 Second-Order Continuous-Time Systems

$$20\log_{10}|H(j\omega)| \cong \begin{cases} 0, & for\,\omega \langle\langle \omega_n \\ -40\log_{10}\omega + 40\log_{10}\omega_n, & for\,\omega \rangle\rangle \omega_n \end{cases}.$$



$$20\log_{10}|H(j\omega)| = -10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2 + 4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}.$$

6.5.2 Second-Order Continuous-Time Systems

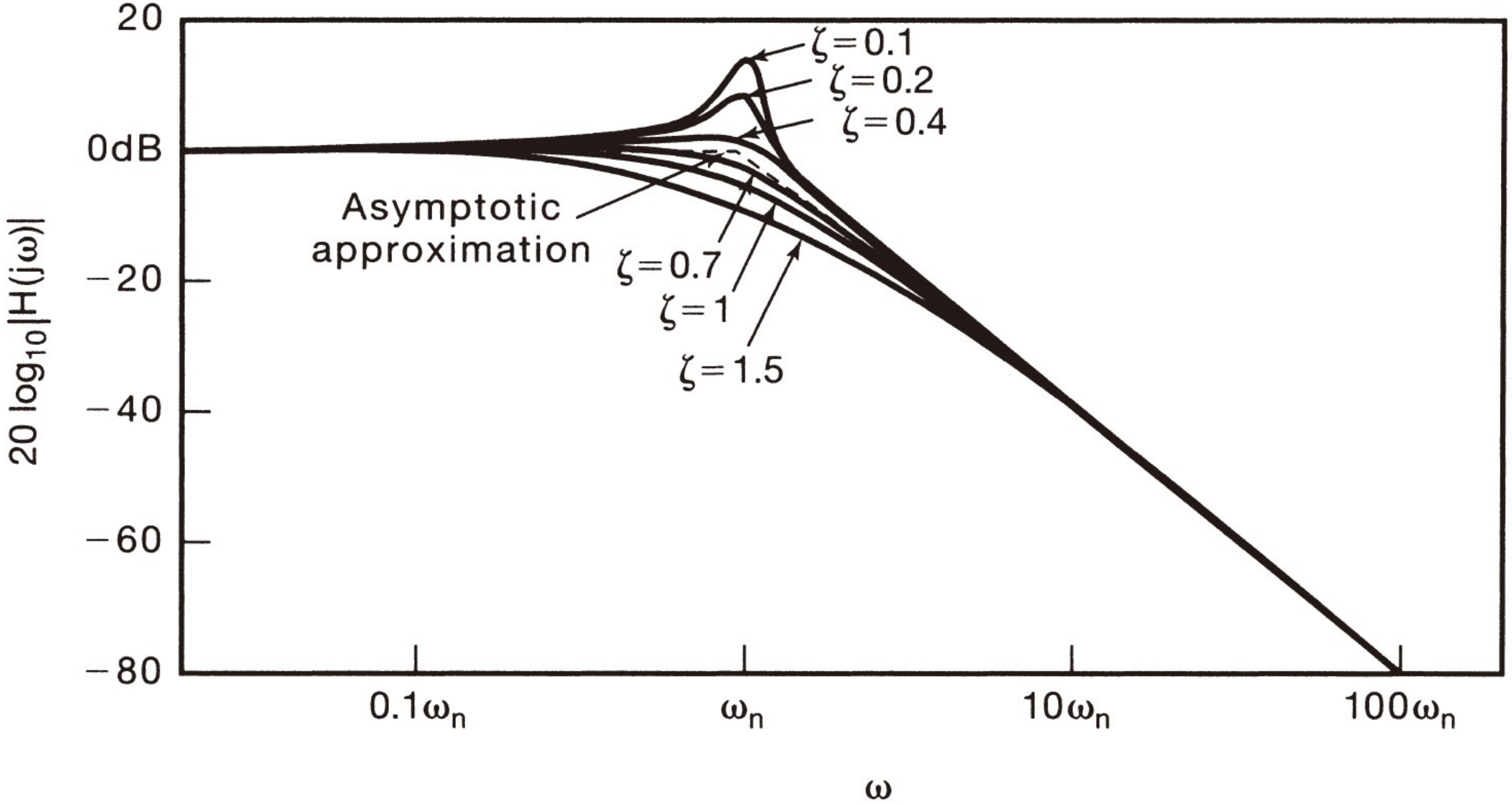
In fact, straightforward calculations using eq. (6.43) show that, for $\zeta < \sqrt{2}/2 \cong 0.707, |H(j\omega)|$ has a maximum value at

$$\omega_{\max} = \omega_n \sqrt{1-2\zeta^2}, \qquad (6.47)$$

and the value at this maximum point is

$$|H(j\omega_{\max})| = \frac{1}{2\zeta\sqrt{1-\zeta^2}}. \qquad (6.48)$$

For $\zeta > 1/\sqrt{2}$ $|H(j\omega)|$, is a monotonically decreasing function



Gain > 1 at specific frequency!

For a second-order circuit described by an equation of the form of eq. (6.31), the **quality** is usually taken to be

$$Q = \frac{1}{2\zeta},$$

6.5.2 Second-Order Continuous-Time Systems

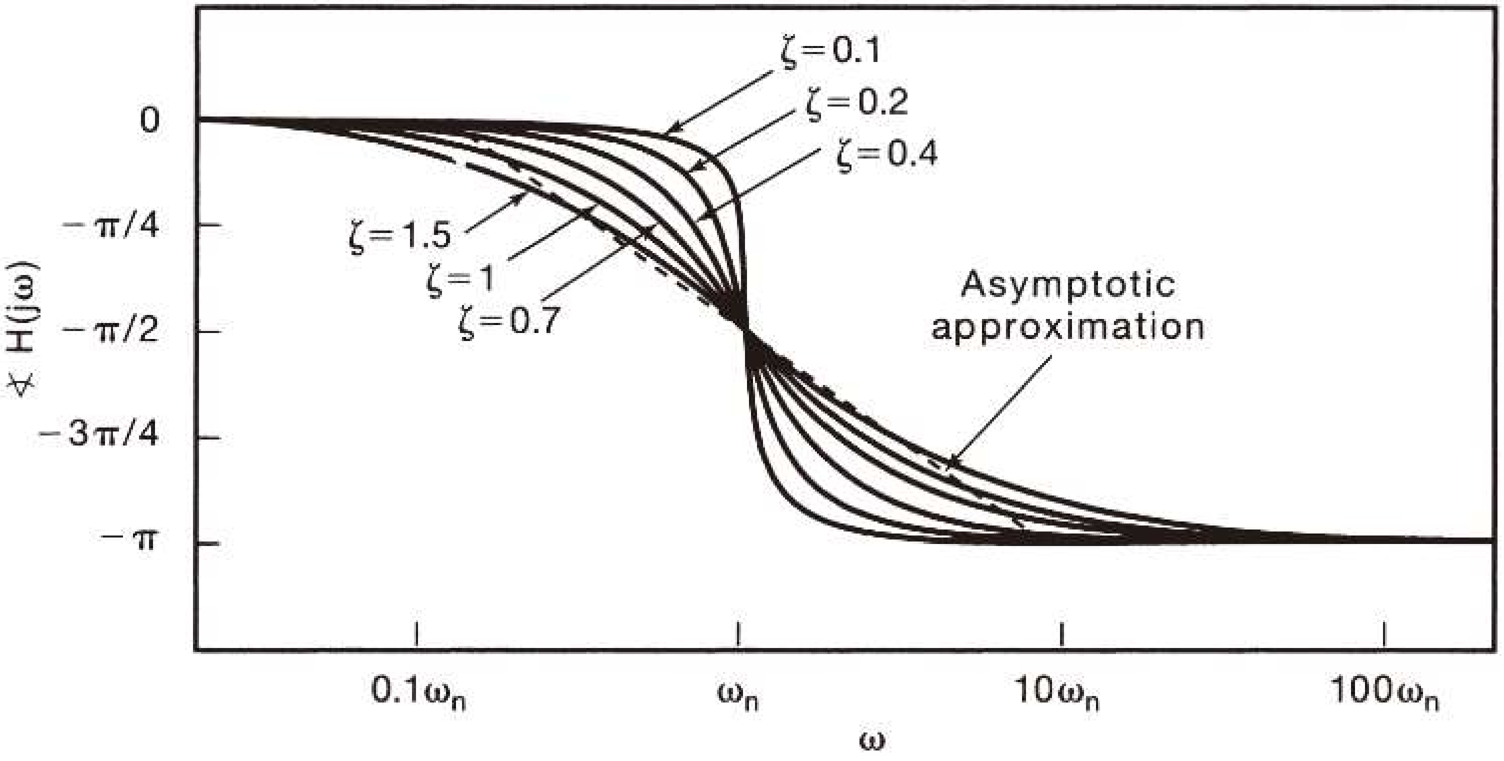
$$H(j\omega) = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1},$$

We can, in addition, obtain a straight-line approximation to $\measuredangle H(j\omega)$, whose exact expression can be obtained from eq. (6.33):

$$\measuredangle H(j\omega) = -\tan^{-1}\left(\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right). \qquad (6.45)$$

The approximation is

$$\measuredangle H(j\omega) \cong \begin{cases} 0, & \omega \le 0.1\omega_n \\ -\dfrac{\pi}{2}\left[\log_{10}\left(\dfrac{\omega}{\omega_n}\right)+1\right], & 0.1\omega_n \le \omega \le 10\omega_n, \\ -\pi, & \omega \ge 10\omega_n \end{cases} \qquad (6.46)$$



$$\measuredangle H(j\omega) \cong \begin{cases} 0, & \omega \le 0.1\omega_n \\ -\dfrac{\pi}{2}\left[\log_{10}\left(\dfrac{\omega}{\omega_n}\right)+1\right], & 0.1\omega_n \le \omega \le 10\omega_n, \\ -\pi, & \omega \ge 10\omega_n \end{cases}$$

Note that the approximation and the actual value again are equal at the break frequency $\omega = \omega_n$, where $\measuredangle H(j\omega_n) = -\dfrac{\pi}{2}.$

6.5.3 Bode Plots for Rational Frequency Responses

1st and 2nd order system can be used as basic building blocks for complex LTI systems with rational frequency responses.

For example,

$$H(j\omega)=\frac{100(1+j\omega)}{(10+j\omega)(100+jw)}$$

$$=(\frac{1}{10})(\frac{1}{1+j\omega/10})(\frac{1}{1+j\omega/100})(1+j\omega)$$

const **1st-order** **numerator**

6.5.3 Bode Plots for Rational Frequency Responses

We can readily obtain the Bode plots for frequency responses of the forms

$$H(j\omega) = 1 + j\omega\tau \qquad (6.49)$$

and

$$H(j\omega) = 1 + 2\zeta\left(\frac{j\omega}{\omega_n}\right) + \left(\frac{j\omega}{\omega_n}\right)^2 . \qquad (6.50)$$

6.5.3 Bode Plots for Rational Frequency Responses

The Bode plots for eqs. (6.49) and (6.50) follow directly from Figures 6.20 and 6.23 and from the fact that

$$20\log_{10}|H(j\omega)| = -20\log_{10}\left|\frac{1}{H(j\omega)}\right|$$

and

$$\sphericalangle\left(H(j\omega)\right) = -\sphericalangle\left(\frac{1}{H(j\omega)}\right).$$

6.5.3 Bode Plots for Rational Frequency Responses

also, consider a system function that is a constant gain

$$H(j\omega) = K.$$

Since $K = |K|e^{j\cdot 0}$ if K > 0 and $K = |K|e^{j\pi}$ if K < 0, we see that

$$20\log_{10}|H(j\omega)| = 20\log_{10}|K|$$

$$\measuredangle H(j\omega) = \begin{cases} 0, & if \quad K\rangle 0 \\ \pi, & if \quad K\langle 0 \end{cases}$$

Example 6.4

Let us obtain the Bode plot for the frequency response

$$H(j\omega) = \frac{2\times 10^4}{(j\omega)^2 + 100j\omega + 10^4}.$$

First, we note that

$$H(j\omega) = 2\hat{H}(j\omega),$$

where $\hat{H}(j\omega)$ has the same form as the standard second-order frequency response specified by eq. (6.33). It follows that

$$20\log_{10}|H(j\omega)| = 20\log_{10}2 + 20\log|\hat{H}(j\omega)|.$$

$$H(j\omega) = \frac{2\times10^4}{(j\omega)^2 + 100 j\omega + 10^4}. \qquad H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2}.$$

By comparing $\hat{H}(j\omega)$ with the frequency response in eq. (6.33), we conclude that $\omega_n = 100$ and ζ = ½ for $\hat{H}(j\omega)$. Using eq. (6.44), we may now specify the asymptotes for $20\log_{10}\left|\hat{H}(j\omega)\right|$:

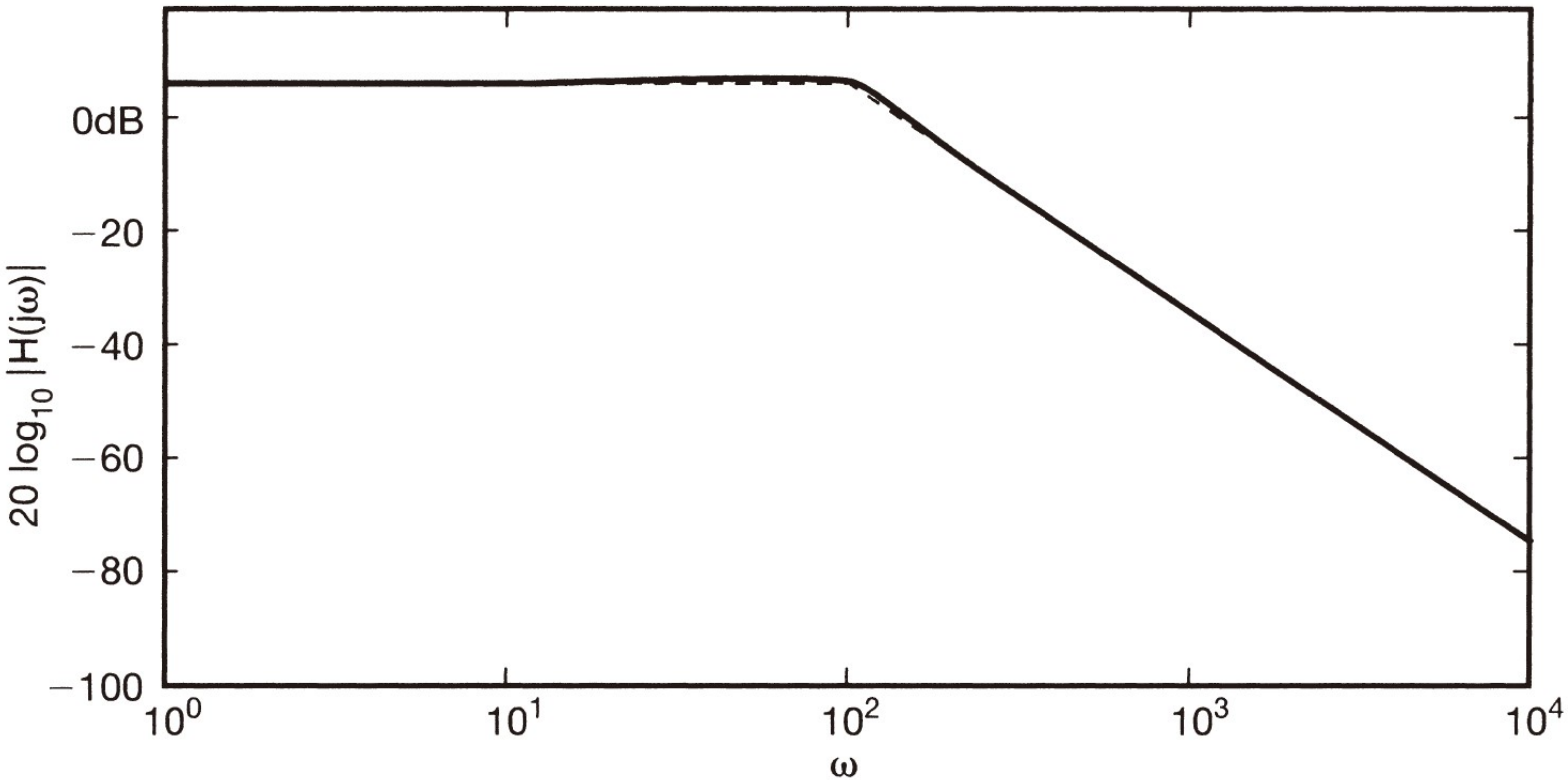
$$20\log_{10}\left|\hat{H}(j\omega)\right| \cong 0 \qquad for \qquad \omega << 100,$$

and

$$20\log_{10}\left|\hat{H}(j\omega)\right| \cong -40\log_{10}\omega + 80 \qquad for \qquad \omega >> 100.$$

$$20\log_{10}\left|H(j\omega)\right| \cong \begin{cases} 0, & for\,\omega \langle\langle \omega_n \\ -40\log_{10}\omega + 40\log_{10}\omega_n, & for\,\omega \rangle\rangle \omega_n \end{cases}.$$

It follows that $20\log_{10}|H(j\omega)|$ will have the same asymptotes, except for a constant offset at all frequencies due to the addition of the $20\log_{10}2$ term (which approximately equals 6 dB). The dashed lines in Figure 6.24(a) represent these asymptotes.



Since the value of ζ for $\hat{H}(j\omega)$ is less than $\sqrt{2}/2$, the actual Bode plot has a slight peak near $\omega = 100$.

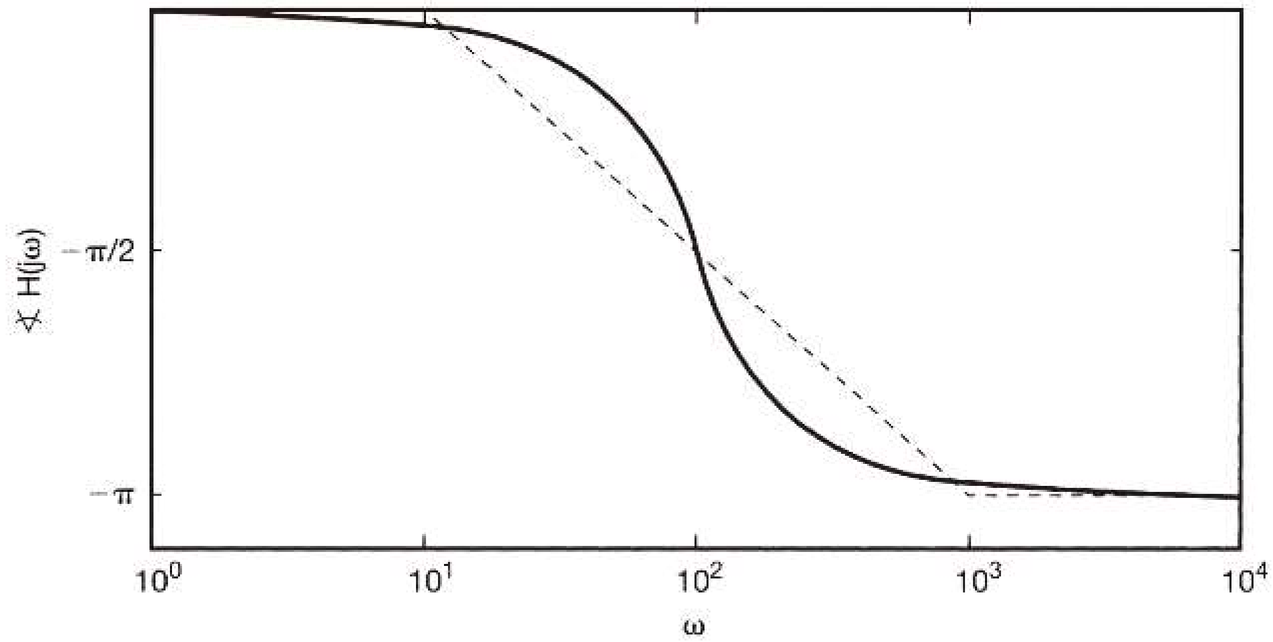
To obtain a plot of $\measuredangle H(j\omega)$, we note that

$$\measuredangle H(j\omega) = \measuredangle \hat{H}(j\omega)$$

and that $\measuredangle \hat{H}(j\omega)$ has its asymptotes specified in accordance with eq. (6.46); that is,

$$\measuredangle \hat{H}(j\omega) = \begin{cases} 0, & \omega \le 10 \\ -(\pi/2)[\log_{10}(\omega/100)+1], & 10 \le \omega \le 1{,}000. \\ -\pi, & \omega \ge 1{,}000. \end{cases}$$

The asymptotes and the actual values for $\measuredangle H(j\omega)$ are plotted with dashed and solid lines.



6.6.1 First-Order Discrete-Time Systems

Consider the first-order causal LTI system described by the difference equation

$$y[n] - ay[n-1] = x[n], \tag{6.51}$$

with $|a| < 1$. the frequency response of this system is

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}, \tag{6.52}$$

and its impulse response is

$$h[n] = a^n u[n], \tag{6.53}$$

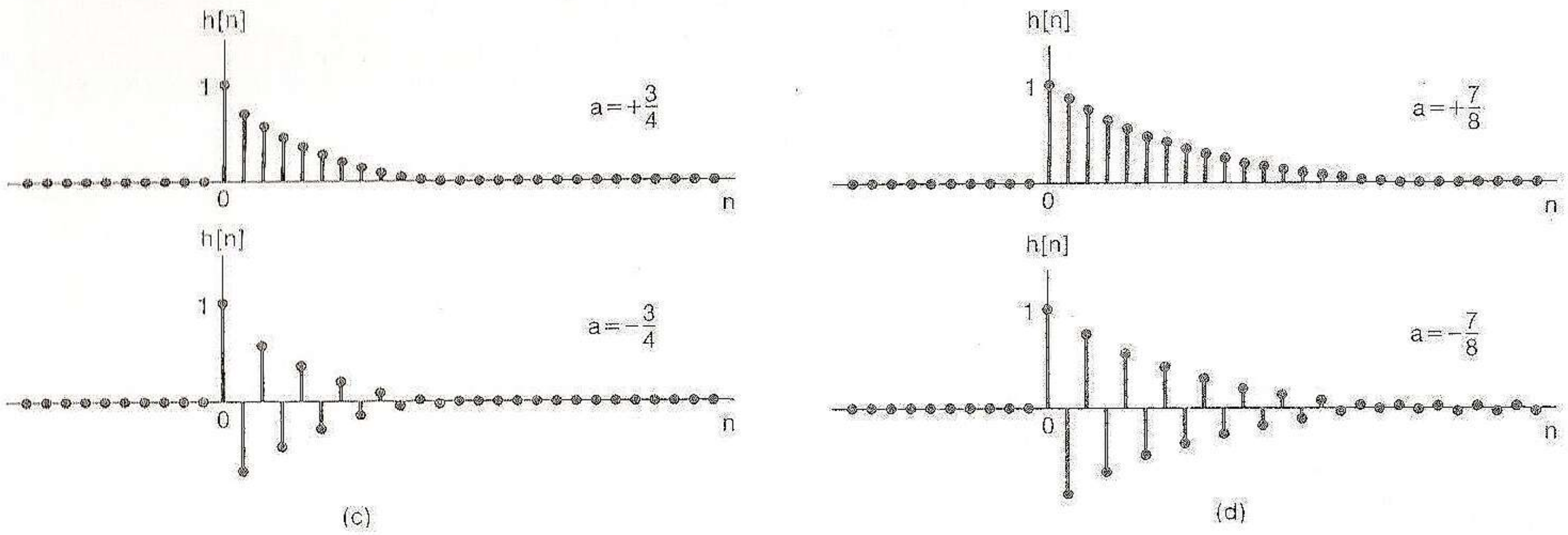
6.6.1 First-Order Discrete-Time Systems

$$h[n] = a^n u[n],$$

Converge faster when $|a|$ is small

Also, the step response of the system is

$$s[n] = h[n] * u[n] = \frac{1-a^{n+1}}{1-a} u[n], \qquad (6.54)$$



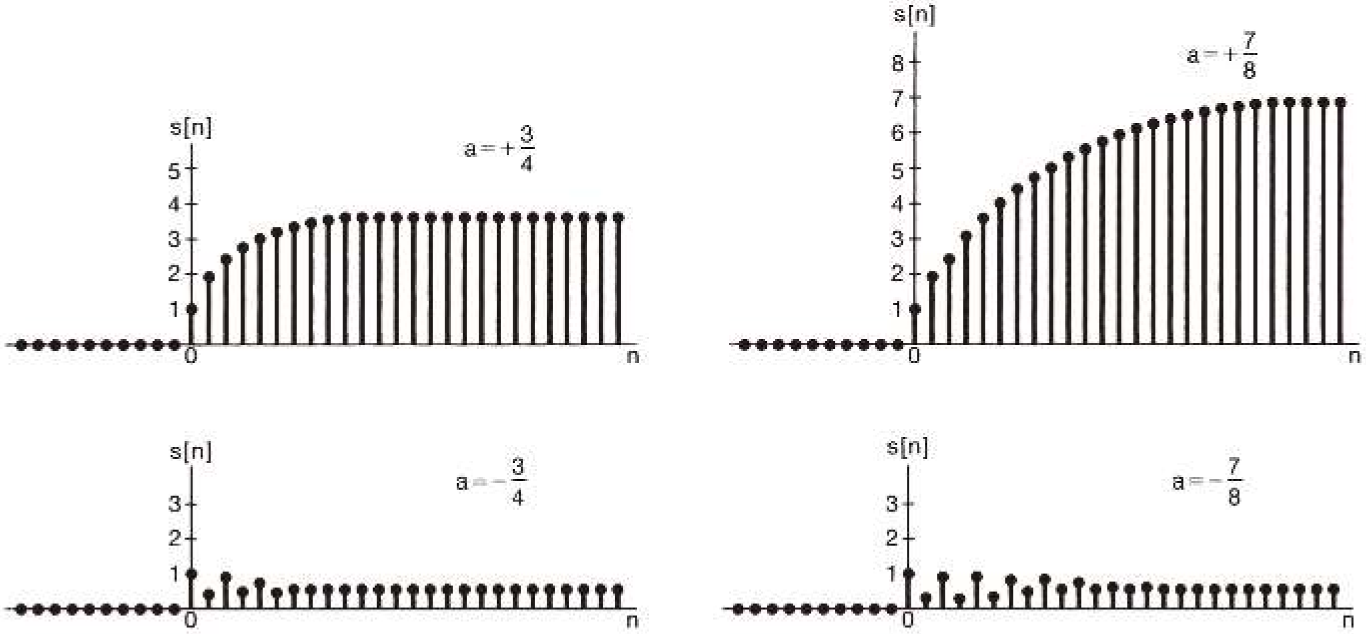
6.6.1 First-Order Discrete-Time Systems

$$h[n] = a^n u[n],$$

Converge faster when $|a|$ is small

Also, the step response of the system is

$$s[n] = h[n] * u[n] = \frac{1-a^{n+1}}{1-a} u[n], \qquad (6.54)$$



6.6.1 First-Order Discrete-Time Systems

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}} = \frac{1}{1 - a\cos\omega + ja\sin\omega},$$

The magnitude and phase of the frequency response of the first-order system in eq. (6.51) are, respectively,

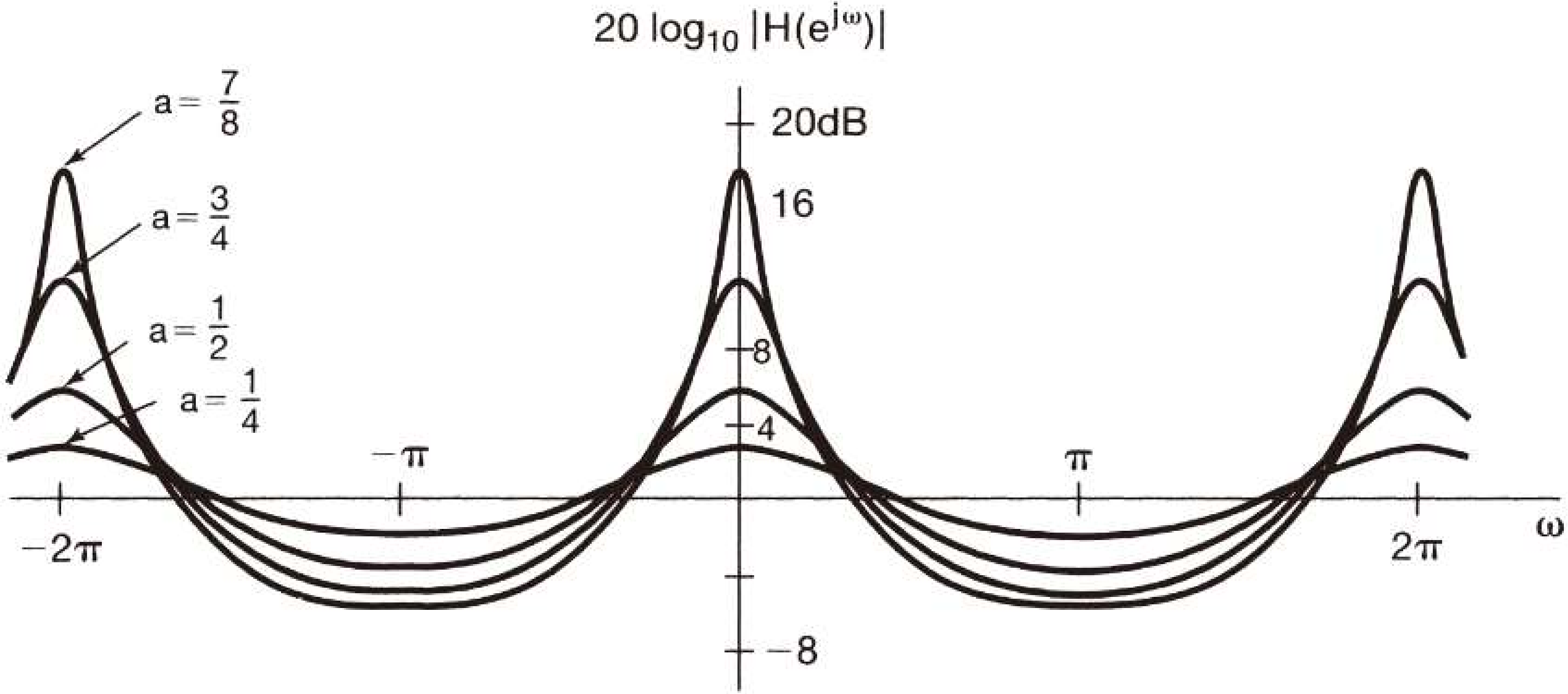
$$\left|H(e^{j\omega})\right| = \frac{1}{(1 + a^2 - 2a\cos\omega)^{1/2}} \qquad (6.55)$$

and

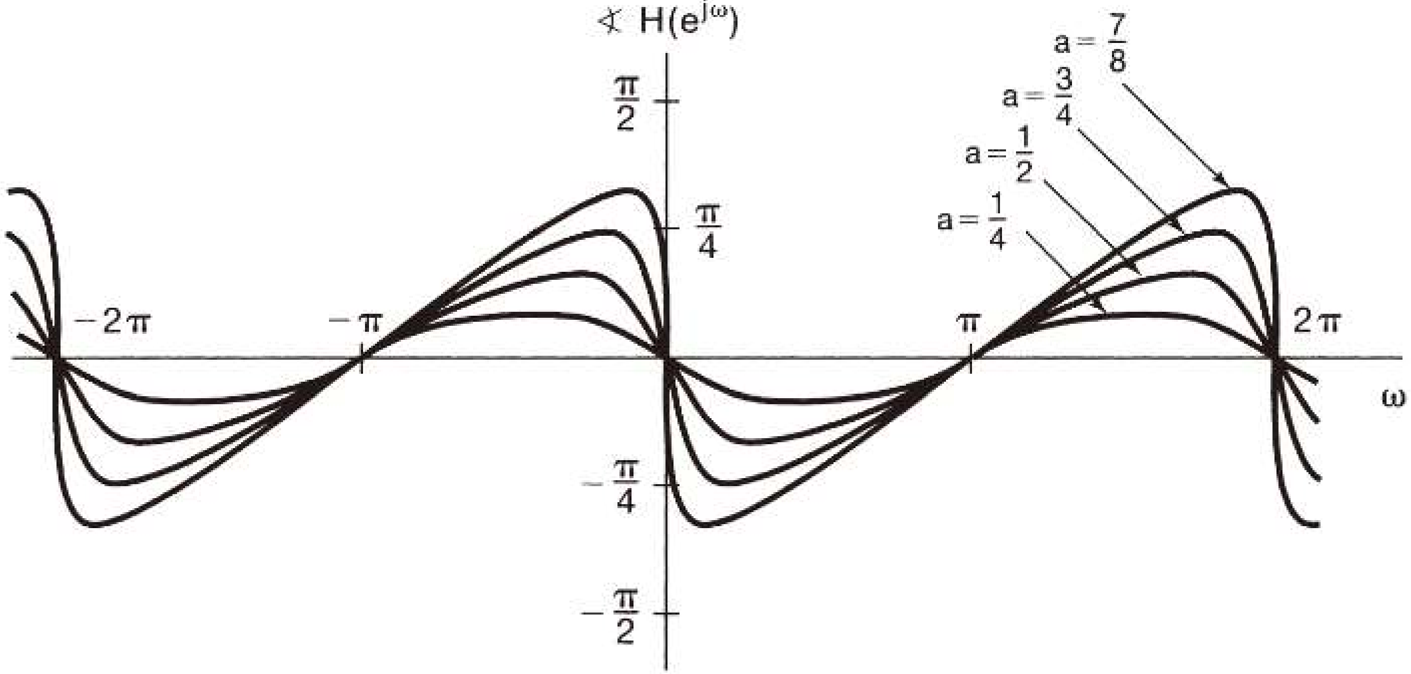
$$\sphericalangle H(e^{j\omega}) = -\tan^{-1}\left[\frac{a\sin\omega}{1 - a\cos\omega}\right]. \qquad (6.56)$$

$$\left|H(e^{j\omega})\right| = \frac{1}{(1+a^2-2a\cos\omega)^{1/2}}$$

For $a > 0$, the system attenuates high frequencies [i.e., $\left|H(e^{j\omega})\right|$ is smaller for ω near $\pm\pi$ than it is for ω near 0] Note also that for $|a|$ small, the maximum and minimum values, 1/(1-a) and 1/(1+a), of $\left|H(e^{j\omega})\right|$ are close together in value, and the graph of $\left|H(e^{j\omega})\right|$ is relatively flat.



$$\sphericalangle H(e^{j\omega}) = -\tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right].$$

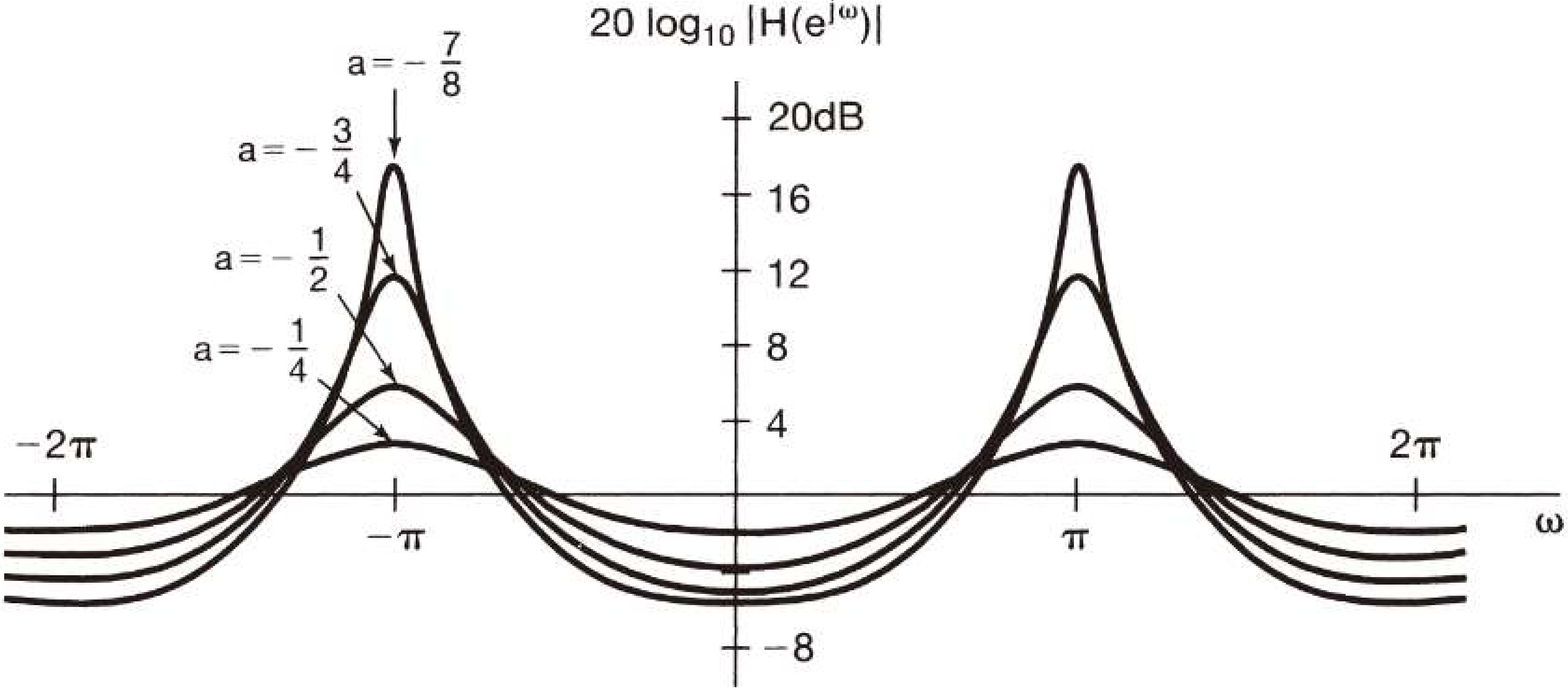


(a)

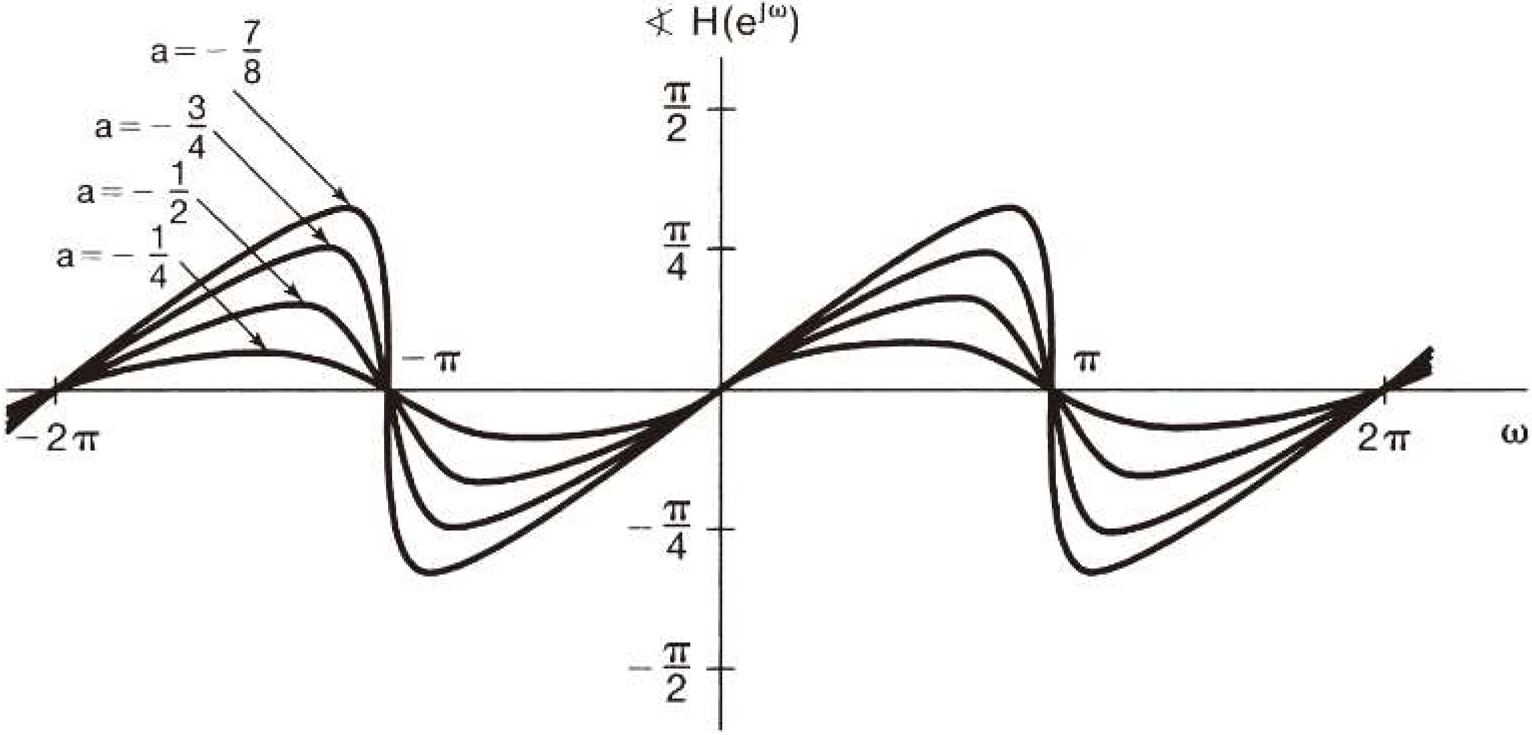
Figure 6.28 Magnitude and phase of the frequency response of eq. (6.52) for a first-order system: (a) plots for several values of $a > 0$; (b) plots for several values of $a < 0$.

$$\left|H(e^{j\omega})\right| = \frac{1}{(1+a^2-2a\cos\omega)^{1/2}}$$

For $a<0$, the system amplifies high frequencies and attenuates low frequencies.



$$\measuredangle H(e^{j\omega}) = -\tan^{-1}\left[\frac{a\sin\omega}{1-a\cos\omega}\right].$$



(b)

Figure 6.28 *Continued*

6.6.2 Second-Order Discrete-Time Systems

Consider next the second-order **causal** LTI system described by

$$y[n]-2r\cos\theta y[n-1]+r^2 y[n-2]=x[n], \qquad (6.57)$$

with $0<r<1$ and $0\leqq\theta\leqq\pi$. The frequency response for this system is

$$H(e^{j\omega})=\frac{1}{1-2r\cos\theta e^{-j\omega}+r^2 e^{-j2\omega}}. \qquad (6.58)$$

$$H(e^{j\omega})=\frac{1}{\left[1-\left(re^{j\theta}\right)e^{-j\omega}\right]\left[1-\left(re^{-j\theta}\right)e^{-j\omega}\right]}.$$

$$H(e^{j\omega}) = \frac{1}{\left[1-\left(re^{j\theta}\right)e^{-j\omega}\right]\left[1-\left(re^{-j\theta}\right)e^{-j\omega}\right]}.$$

For $\theta \neq 0$ or π, the two factors in the denominator of $H(e^{j\omega})$ are different, and a partial-fraction expansion yields

$$H(e^{j\omega}) = \frac{A}{1-(re^{j\theta})e^{-j\omega}} + \frac{B}{1-(re^{-j\theta})e^{-j\omega}}, \qquad (6.60)$$

where

$$A = \frac{e^{j\theta}}{2j\sin\theta}, \qquad B = \frac{e^{-j\theta}}{2j\sin\theta}. \qquad (6.61)$$

In this case, the impulse response of the system is

$$h[n] = \left[A(re^{j\theta})^n + B(re^{-j\theta})^n\right]u[n] \qquad (6.62)$$

$$= r^n \frac{\sin[(n+1)\theta]}{\sin\theta}u[n].$$

6.6.2 Second-Order Discrete-Time Systems

$(n+1)a^n u[n], \quad \|a\| < 1$	$\dfrac{1}{(1-ae^{-j\omega})^2}$

For $\theta = 0$ or π, the two factors in the denominator of eq. (6.58) are the same. When $\theta = 0$,

$$H(e^{j\omega}) = \frac{1}{\left(1 - re^{-j\omega}\right)^2} \tag{6.63}$$

and

$$h[n] = (n+1)r^n u[n]. \tag{6.64}$$

When $\theta = \pi$,

$$H(e^{j\omega}) = \frac{1}{\left(1 + re^{-j\omega}\right)^2} \tag{6.65}$$

and

$$h[n] = (n+1)(-r)^n u[n]. \tag{6.66}$$

$$h[n] = \left[A(re^{j\theta})^n + B(re^{-j\theta})^n\right]u[n]$$

For $\theta \neq 0$ or π

$$s[n] = h[n] * u[n] = \sum_{m=0}^{n} h[m] = \left[A\left(\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}}\right) + B\left(\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right)\right]u[n].$$

(6.67)

Also, using the result of Problem 2.52, we find that for $\theta = 0$,

$$s[n] = \left[\frac{1}{(r-1)^2} - \frac{r}{(r-1)^2}r^n + \frac{r}{r-1}(n+1)r^n\right]u[n], \quad (6.68)$$

while for $\theta = \pi$,

$$s[n] = \left[\frac{1}{(r+1)^2} + \frac{r}{(r+1)^2}(-r)^n + \frac{r}{r+1}(n+1)(-r)^n\right]u[n]. \quad (6.69)$$

6.6.2 Second-Order Discrete-Time Systems

$$h[n] = \left[A(re^{j\theta})^n + B(re^{-j\theta})^n\right]u[n]$$

$$= r^n \frac{\sin[(n+1)\theta]}{\sin\theta} u[n].$$

$\theta = 0;$

$$h[n] = (n+1)r^n u[n].$$

$\theta = \pi;$

$$h[n] = (n+1)(-r)^n u[n].$$

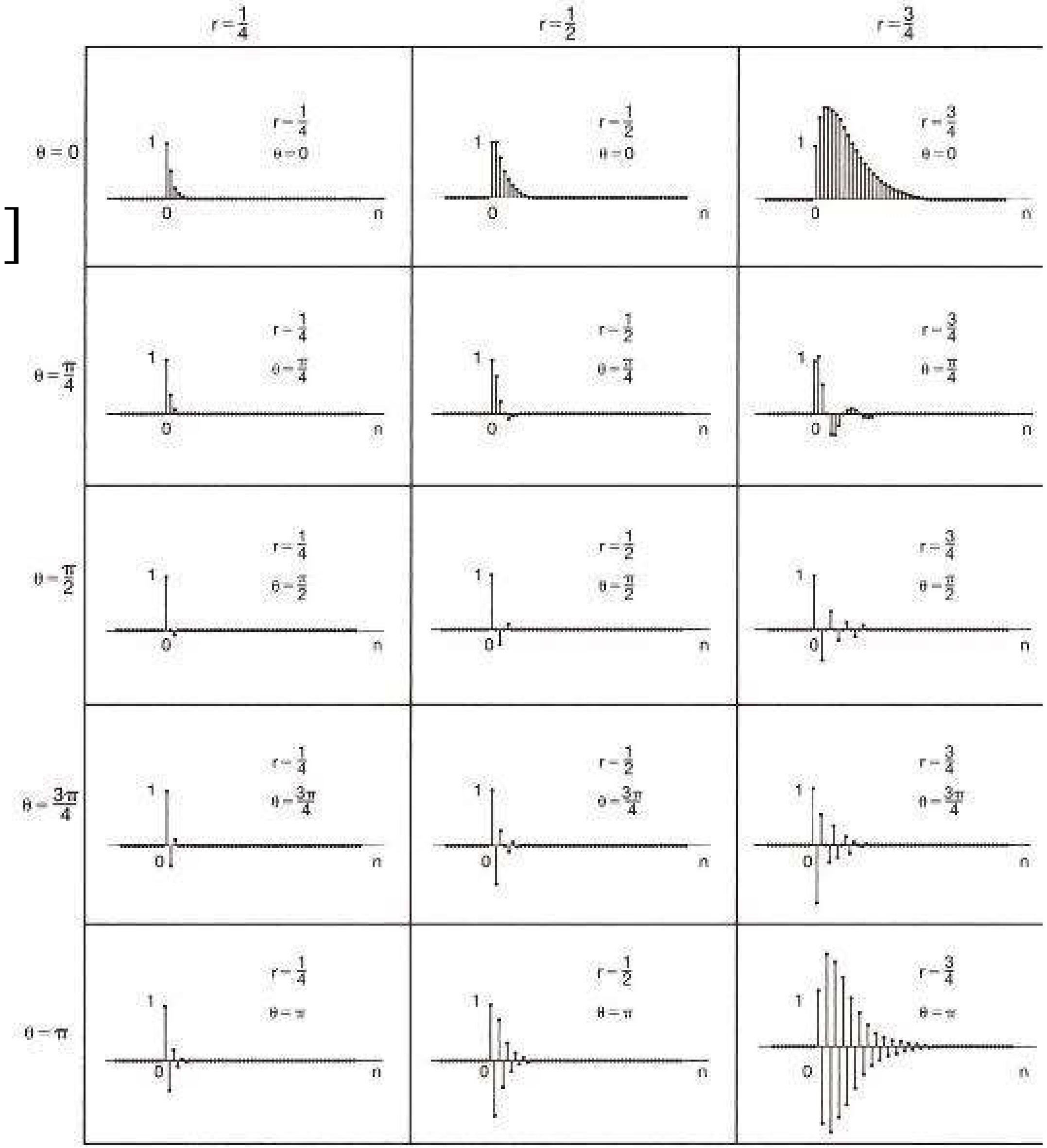


Figure 6.29 Impulse response of the second-order system of eq. (6.57) for a range of values of r and θ. 圖 6.29 為在不同的 r 和 θ 值下的脈衝響應。

6.6.2 Second-Order Discrete-Time Systems

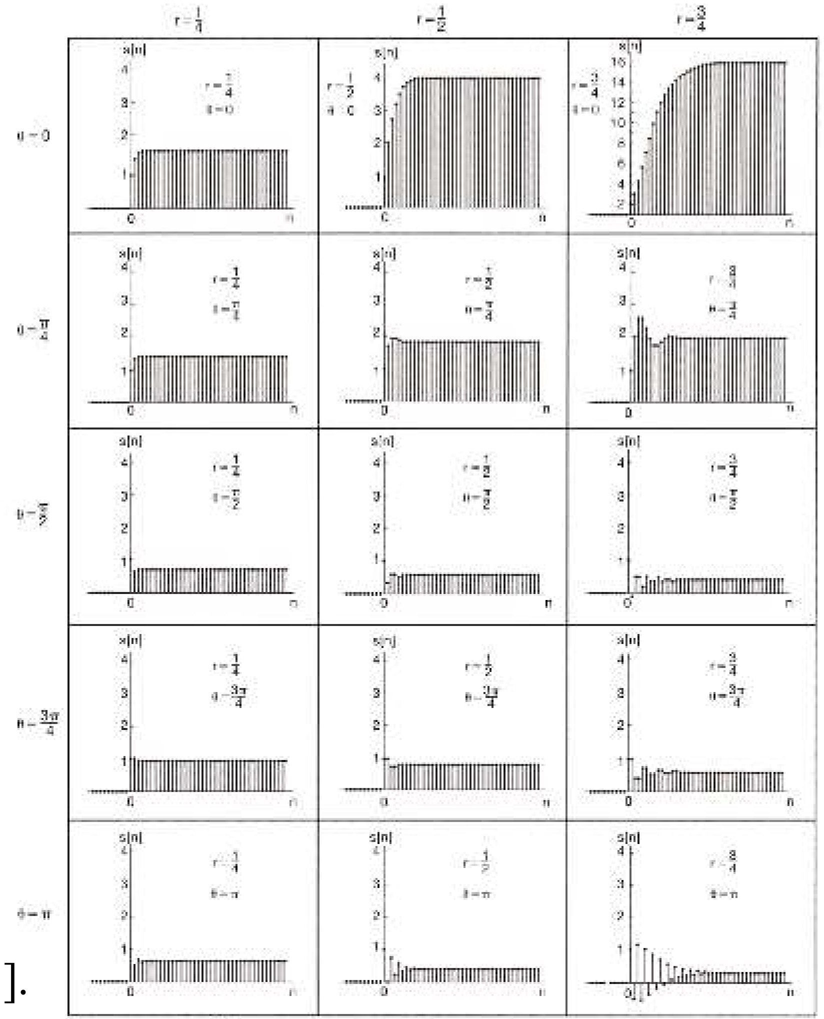
$$s[n]=\left[A\left(\frac{1-\left(re^{j\theta}\right)^{n+1}}{1-re^{j\theta}}\right)+B\left(\frac{1-\left(re^{-j\theta}\right)^{n+1}}{1-re^{-j\theta}}\right)\right]u[n].$$

$\theta = 0;$

$$s[n]=\left[\frac{1}{(r-1)^2}-\frac{r}{(r-1)^2}r^n+\frac{r}{r-1}(n+1)r^n\right]u[n],$$

$\theta = \pi;$

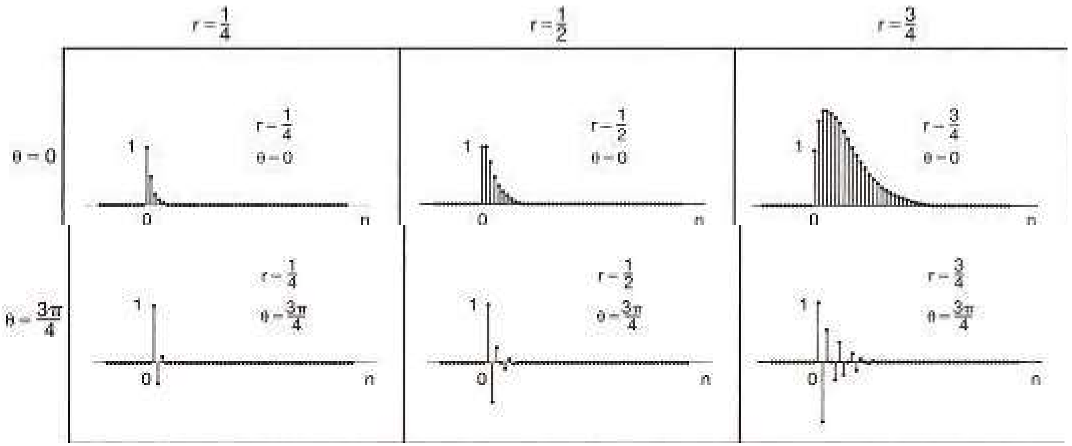
$$s[n]=\left[\frac{1}{(r+1)^2}+\frac{r}{(r+1)^2}(-r)^n+\frac{r}{r+1}(n+1)(-r)^n\right]u[n].$$

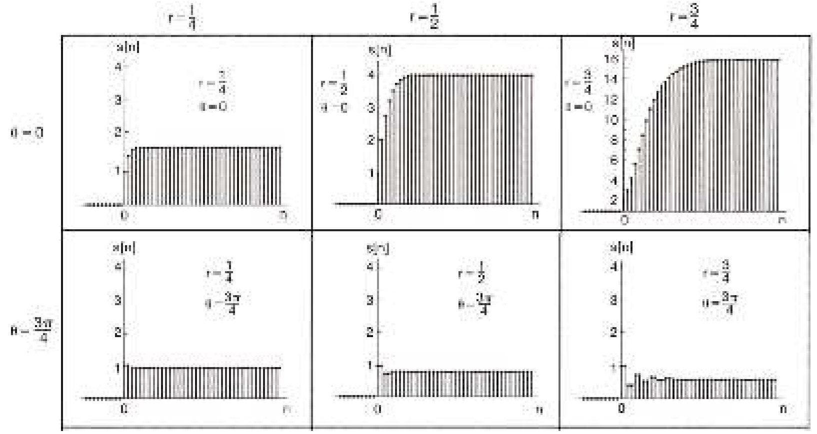


* Note: The plot for $r=\frac{3}{4}$, $\theta=0$ has a different scale from the others.

Figure 6.30 Step response of the second-order system of eq. (6.57) for a range of values of r and θ. 圖 6.30 為在不同的 r 和 θ 值下的步級響應。

6.6.2 Second-Order Discrete-Time Systems

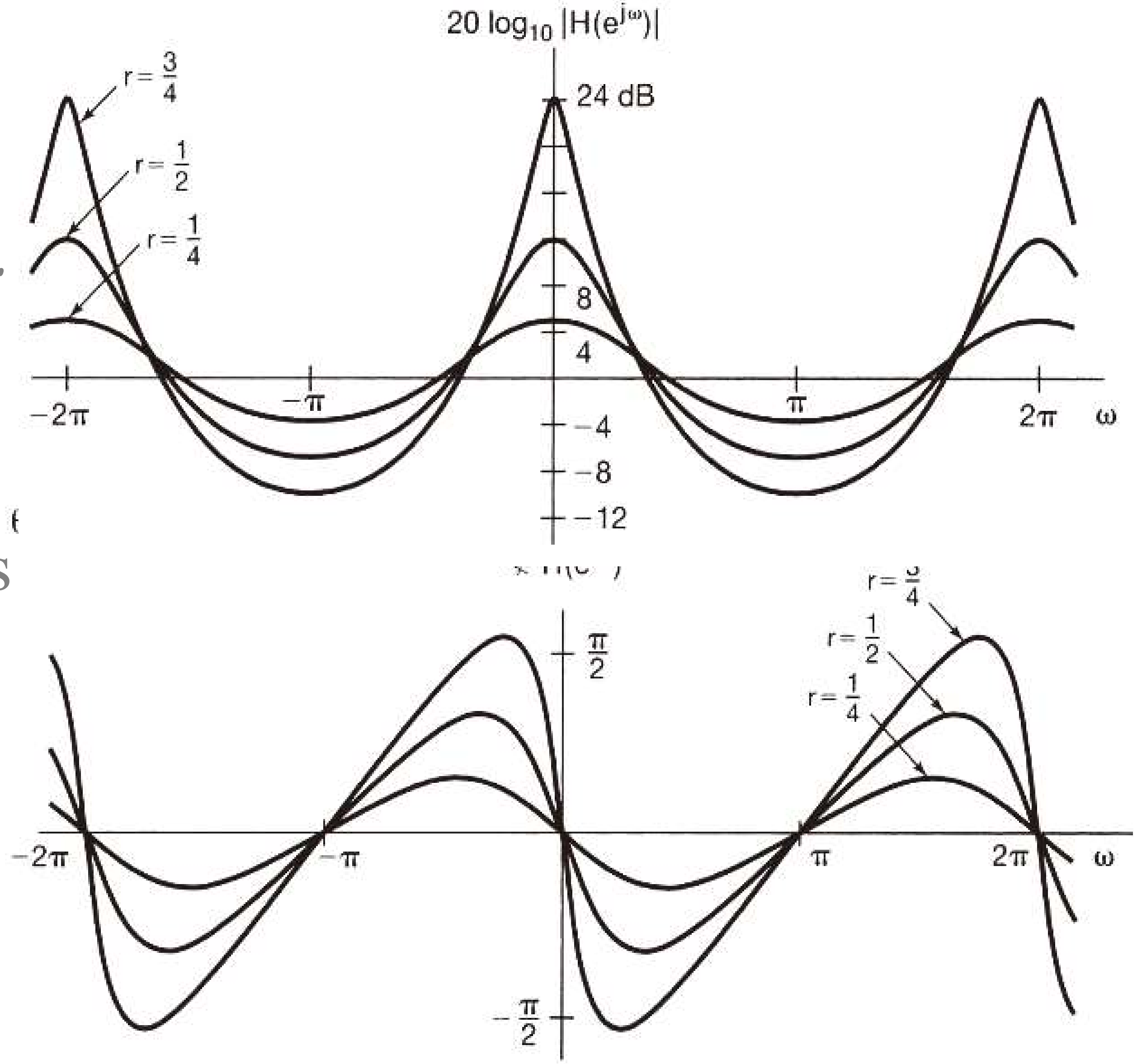




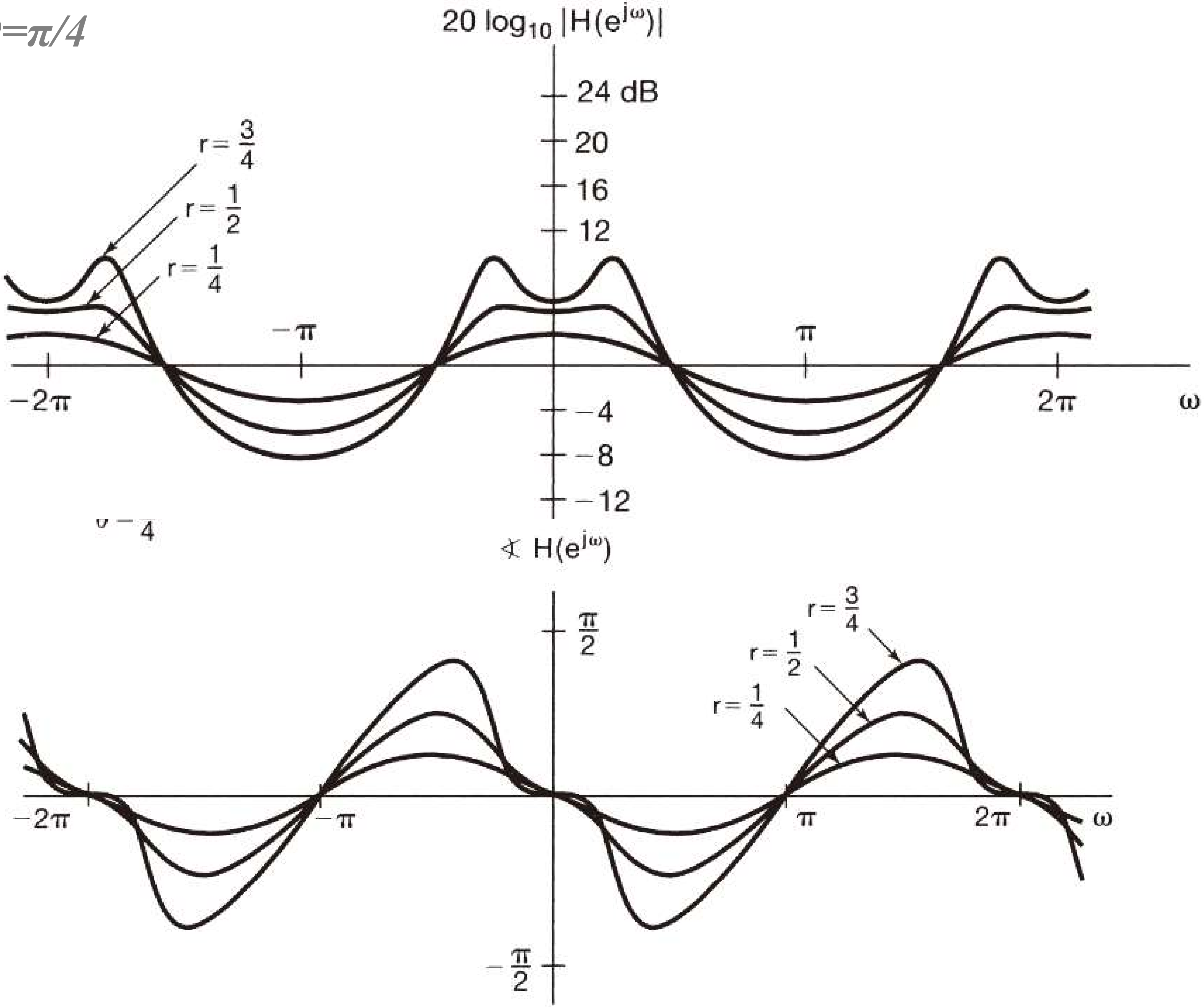
The second-order system given is the counterpart of the **underdamped** second-order system in continuous time, while the special case of $\theta = 0$ is the critically damped case. For any value of $\theta \neq 0$, the impulse response has a damped ocillatory behavior, and the step response exhibits ringing and overshoots.

The frequency response of this system is depicted in Figure 6.31 for a number of values of r and θ. From Figure 6.31, we see that a band of frequencies is amplified, and r determines how sharply peaked the frequency response is within this band.

θ=0

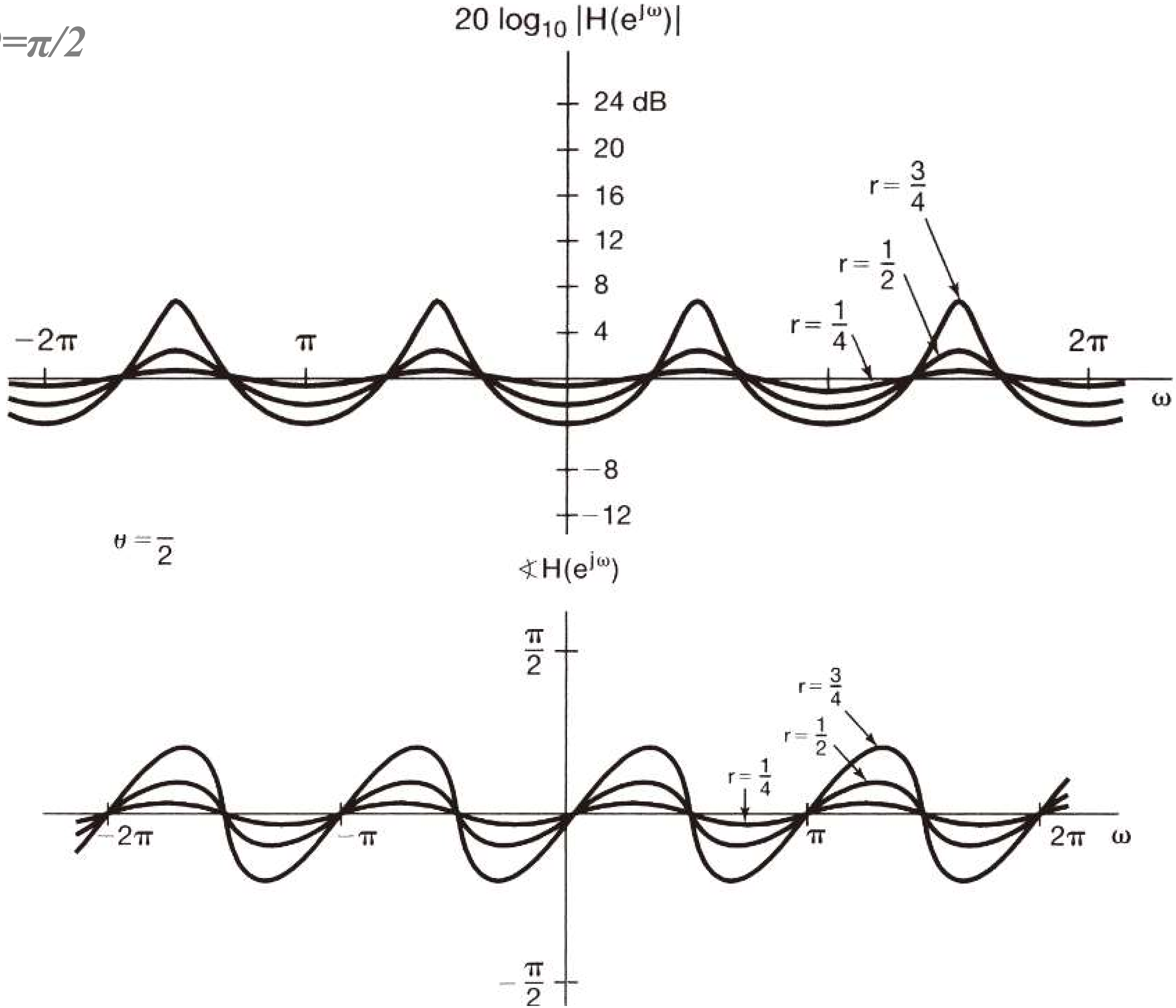


Θ=π/4



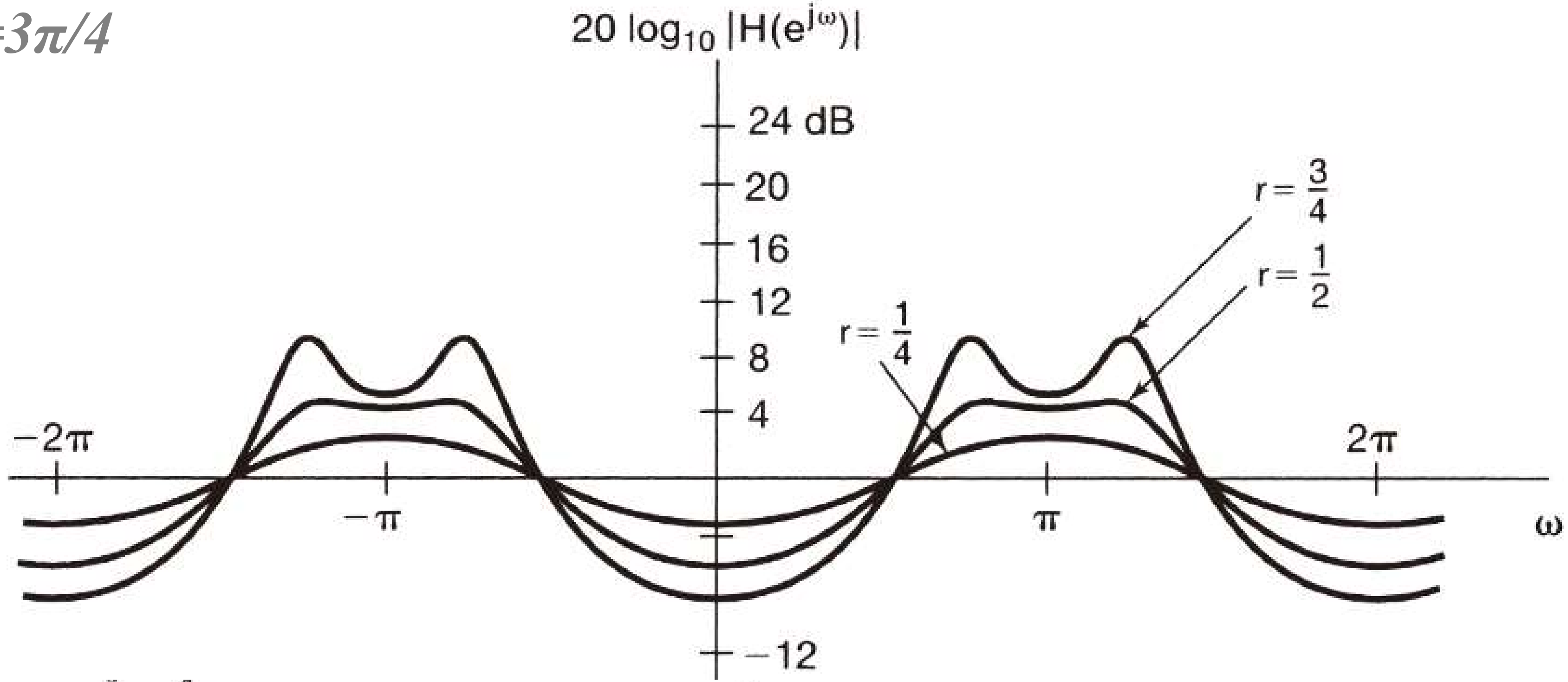
20 log10 |H(e^jω)|
24 dB
20
16
12
−4
−8
−12
r = 3/4
r = 1/2
r = 1/4
−2π
−π
π
2π
ω
∢ H(e^jω)
π/2
−π/2
r = 3/4
r = 1/2
r = 1/4
−2π
−π
π
2π
ω

$\Theta=\pi/2$

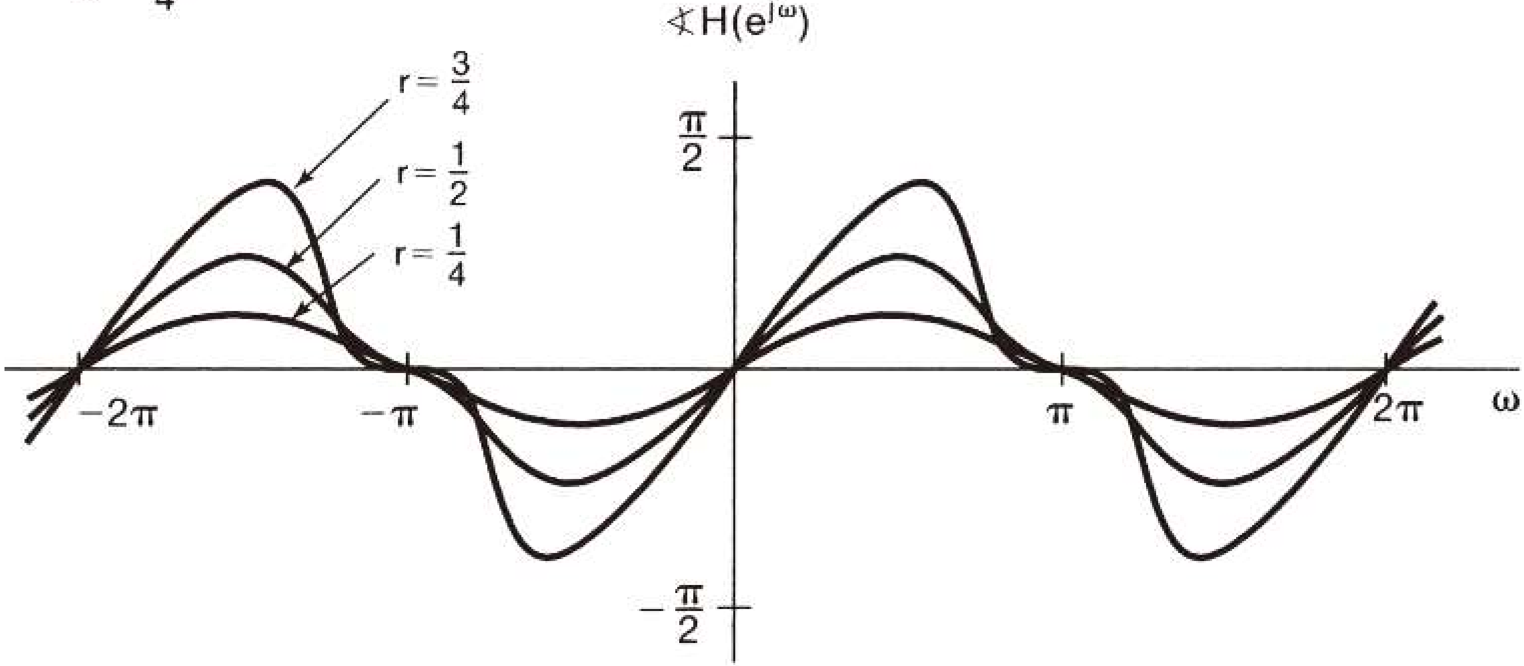


20 log10 |H(e^jω)|
24 dB
20
16
12
8
4
−8
−12
−2π
π
2π
ω
r = 3/4
r = 1/2
r = 1/4
θ = π/2
∢H(e^jω)
π/2
−π/2
−2π
−π
π
2π
ω
r = 3/4
r = 1/2
r = 1/4

Θ=3π/4

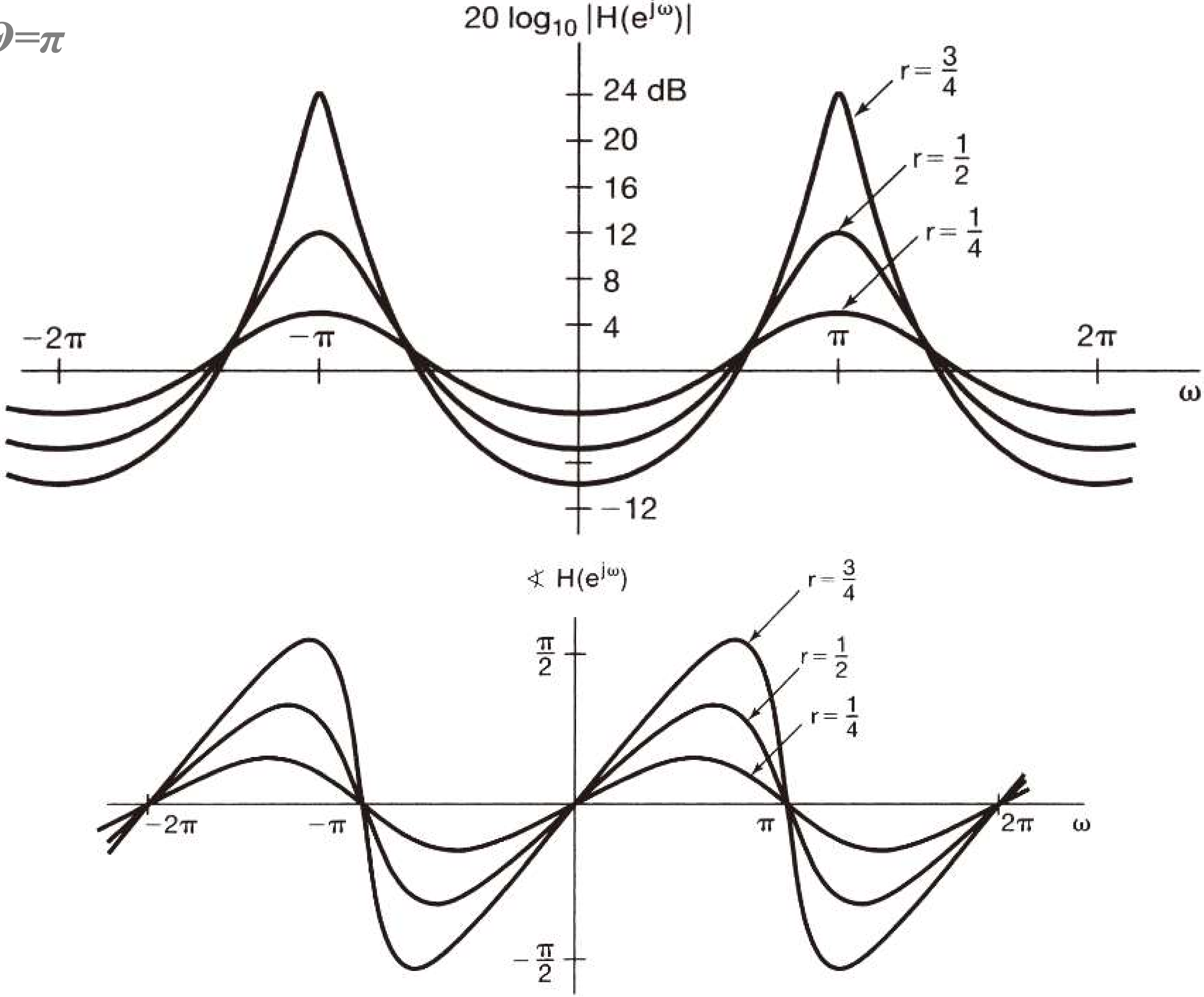


20 log10 |H(e^jω)|
24 dB
20
16
12
8
4
−12
−2π
−π
π
2π
ω
r = 3/4
r = 1/2
r = 1/4



∢H(e^jω)
π/2
−π/2
−2π
−π
π
2π
ω
r = 3/4
r = 1/2
r = 1/4

$\Theta=\pi$



20 log10 |H(e^jω)|
24 dB
20
16
12
8
4
−12
−2π
−π
π
2π
ω
r = 3/4
r = 1/2
r = 1/4
∢ H(e^jω)
π/2
−π/2
−2π
−π
π
2π
ω
r = 3/4
r = 1/2
r = 1/4

6.6.2 Second-Order Discrete-Time Systems

$$H(e^{j\omega}) = \frac{1}{\left[1-\left(re^{j\theta}\right)e^{-j\omega}\right]\left[1-\left(re^{-j\theta}\right)e^{-j\omega}\right]}.$$

It is also possible to consider second-order systems having factors with **real coefficients**. Specifically, consider

$$H(e^{j\omega}) = \frac{1}{\left(1-d_1e^{-j\omega}\right)\left(1-d_2e^{-j\omega}\right)} \quad (6.70)$$

$$= \frac{1}{1-(d_1+d_2)e^{-j\omega}+d_1d_2e^{-j2\omega}}$$

$$\Downarrow$$

$$y[n]-\left(d_1+d_2\right)y[n-1]+d_1d_2y[n-2]=x[n] \quad (6.71)$$

6.6.2 Second-Order Discrete-Time Systems

$$H(e^{j\omega}) = \frac{1}{\left(1 - d_1 e^{-j\omega}\right)\left(1 - d_2 e^{-j\omega}\right)}$$

In this case,

$$H(e^{j\omega}) = \frac{A}{1 - d_1 e^{-j\omega}} + \frac{B}{1 - d_2 e^{-j\omega}}, \tag{6.72}$$

where

$$A = \frac{d_1}{d_1 - d_2}, \qquad B = \frac{d_2}{d_2 - d_1}. \tag{6.73}$$

Thus,

$$h[n] = \left[A d_1^n + B d_2^n\right] u[n], \tag{6.74}$$

6.7.1 Automobile Suspension Systems

汽車懸吊系統的機
械元件結構示意圖

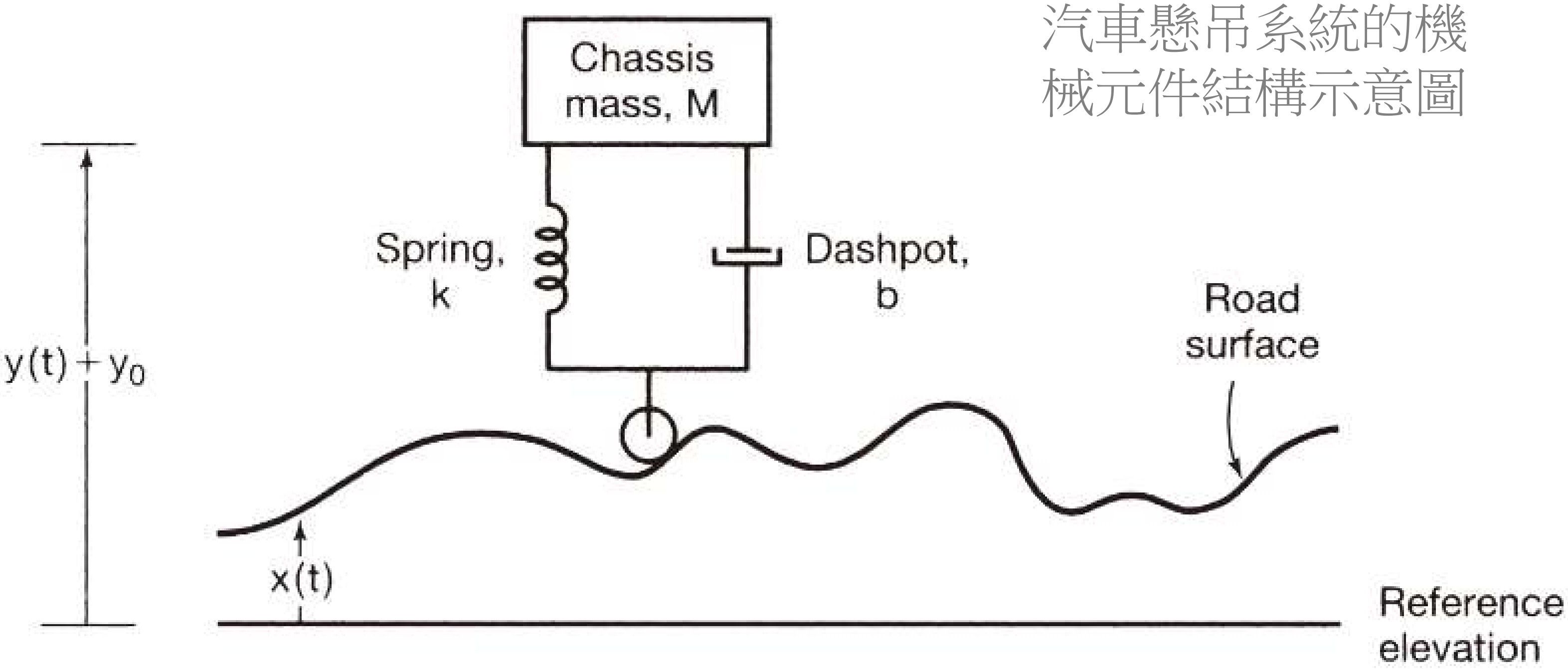


Figure 6.32 Diagrammatic representation of an automotive suspension system. Here, y_0 represents the distance between the chassis and the road surface when the automobile is at rest, $y(t) + y_0$ the position of the chassis above the reference elevation, and $x(t)$ the elevation of the road above the reference elevation.

k and b are the spring and shock absorber constants

$$H(j\omega) = \frac{\omega_n^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{1}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1},$$

The differential equation governing the motion of the chassis is then

$$M\frac{d^2y(t)}{dt^2} + b\frac{dy(t)}{dt} + ky(t) = kx(t) + b\frac{dx(t)}{dt}, \quad (6.76)$$

$$\omega_n = \sqrt{\frac{k}{M}}$$

For fix mass, spring constant controls natural frequency

$$H(j\omega) = \frac{k + bj\omega}{(j\omega)^2 M + b(j\omega) + k},$$

$$2\zeta\omega_n = \frac{b}{M}$$

For fix mass and fix k, shock absorber constant controls damping ratio

$$H(j\omega) = \frac{\omega_n^2 + 2\zeta\omega_n(j\omega)}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n^2} = \frac{(1 + 2\zeta(j\omega/\omega_n))}{(j\omega/\omega_n)^2 + 2\zeta(j\omega/\omega_n) + 1} (6,77)$$

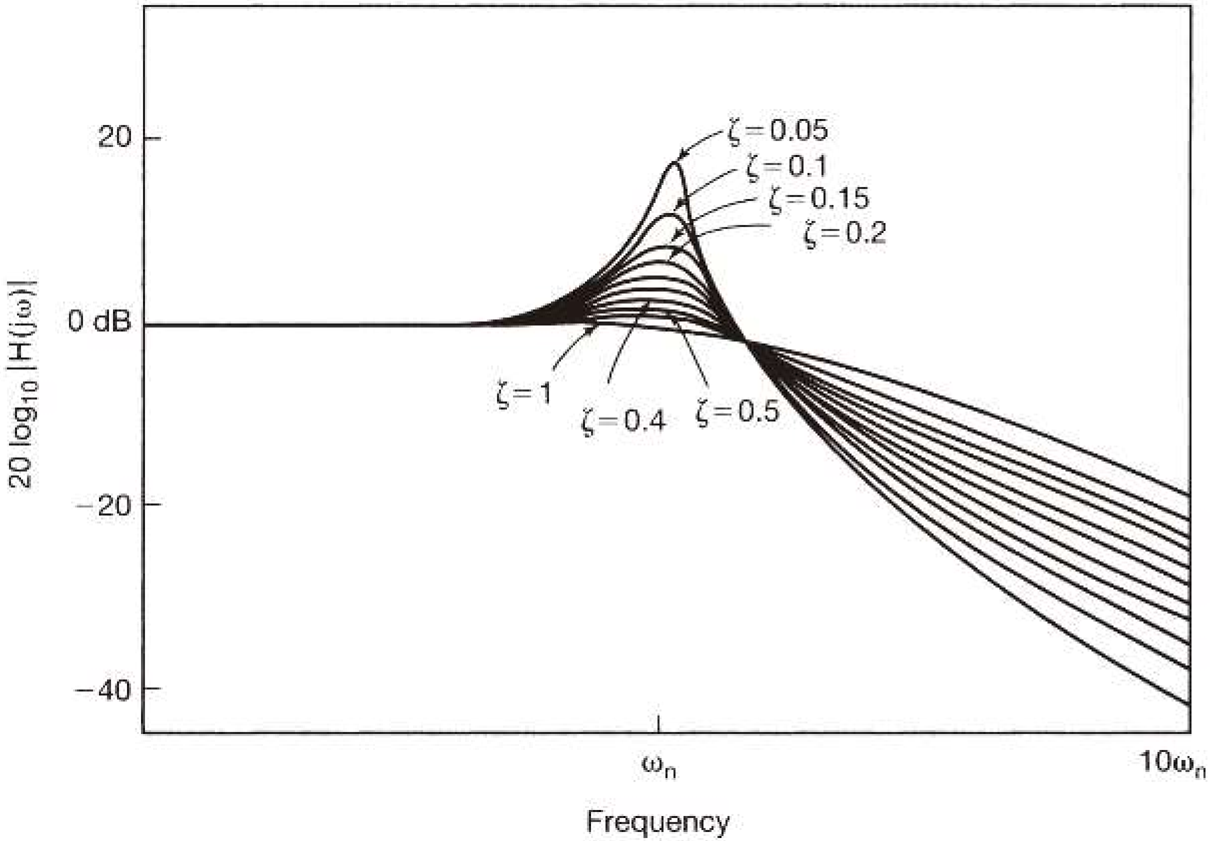


Figure 6.33 Bode plot for the magnitude of the frequency response of the automobile suspension system for several values of the damping ratio.

We want ω_n to be small so that the high frequency road fluctuation can be filtered and the ride becomes smoother.

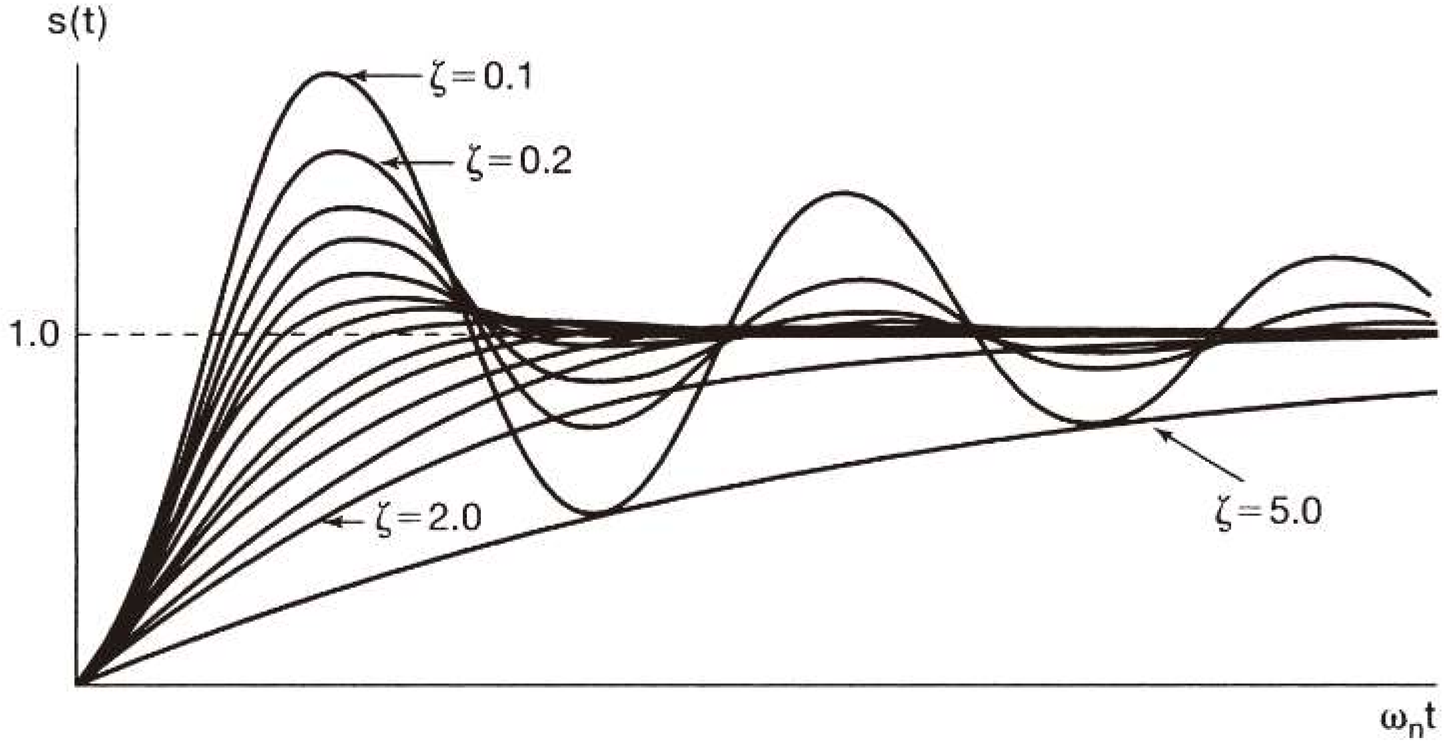


Figure 6.34 Step response of the automotive suspension system for various values of the damping ratio (ζ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0, 5.0).

But when ω_n decreases, the rise time increases, since $\omega_n t$ is fixed for fix ζ. This will make our system responses slowly.
This shows that there is a time vs frequency domain trade-off.

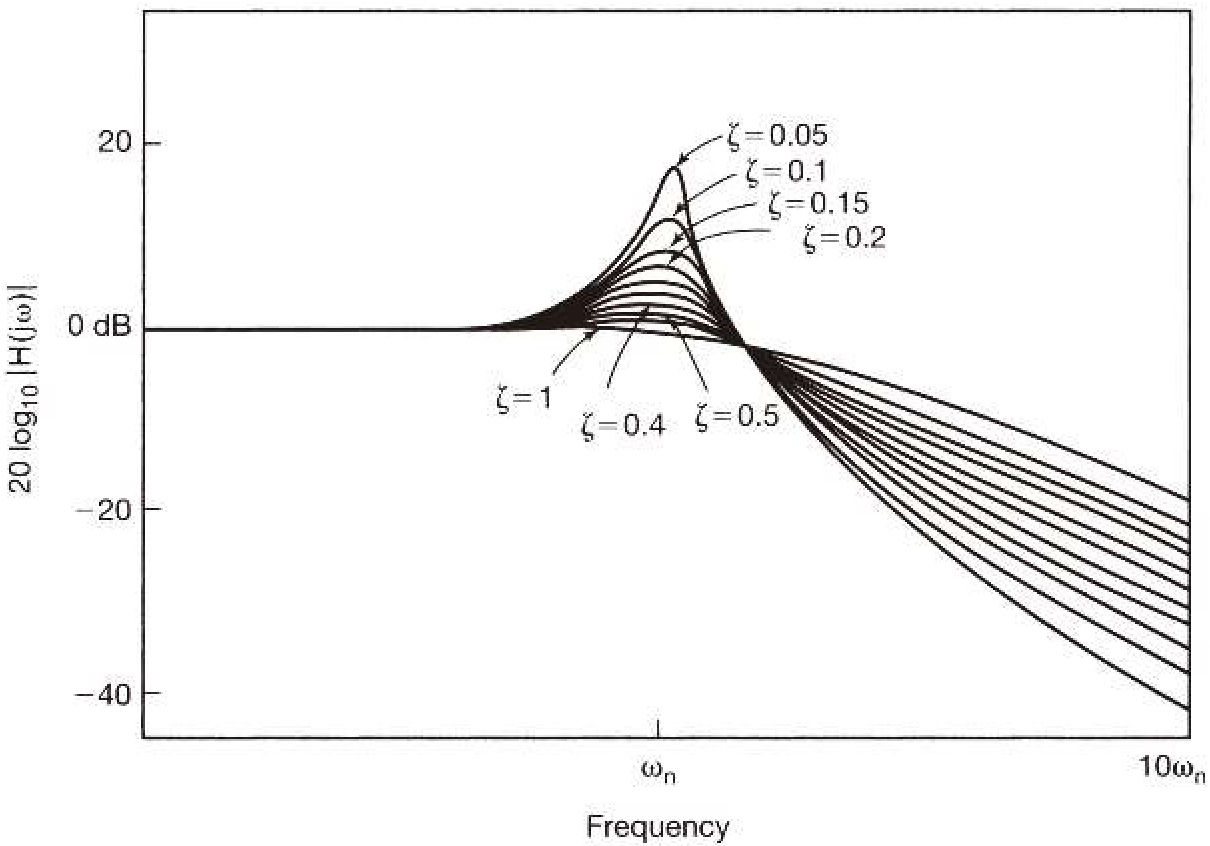


Figure 6.33 Bode plot for the magnitude of the frequency response of the automobile suspension system for several values of the damping ratio.

We want ζ to be small so that the low-pass filter cut-off frequency sharply.

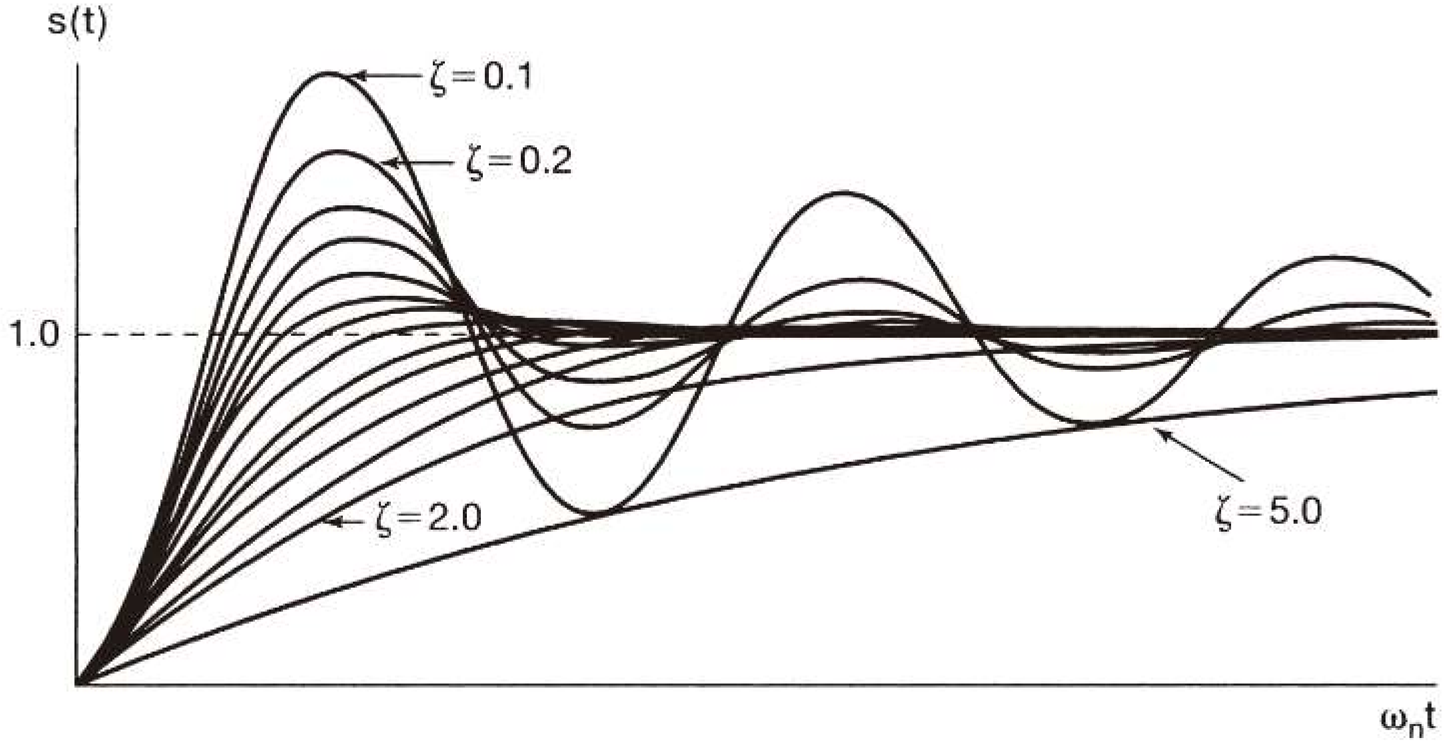
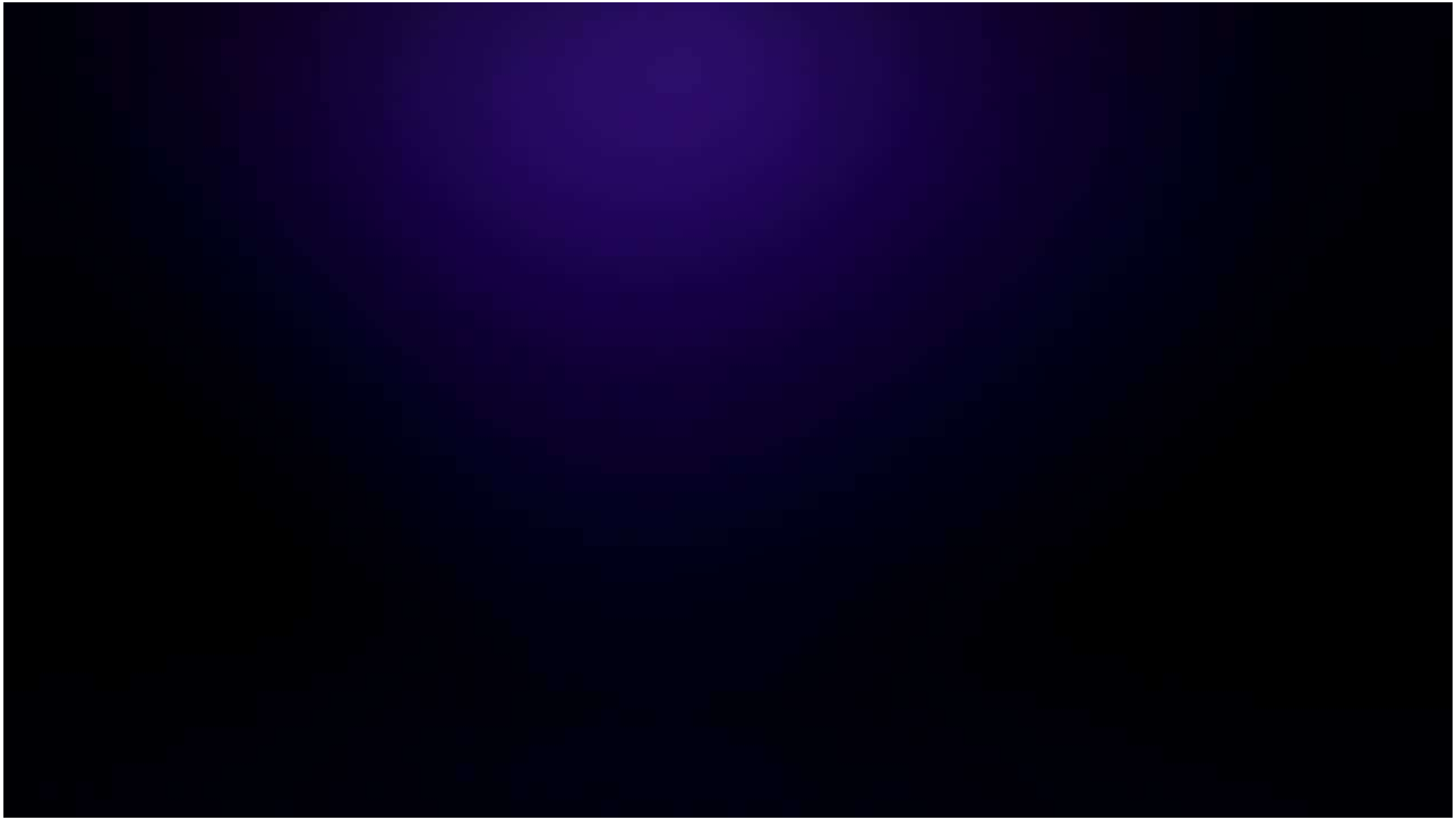


Figure 6.34 Step response of the automotive suspension system for various values of the damping ratio (ζ = 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0, 1.2, 1.5, 2.0, 5.0).

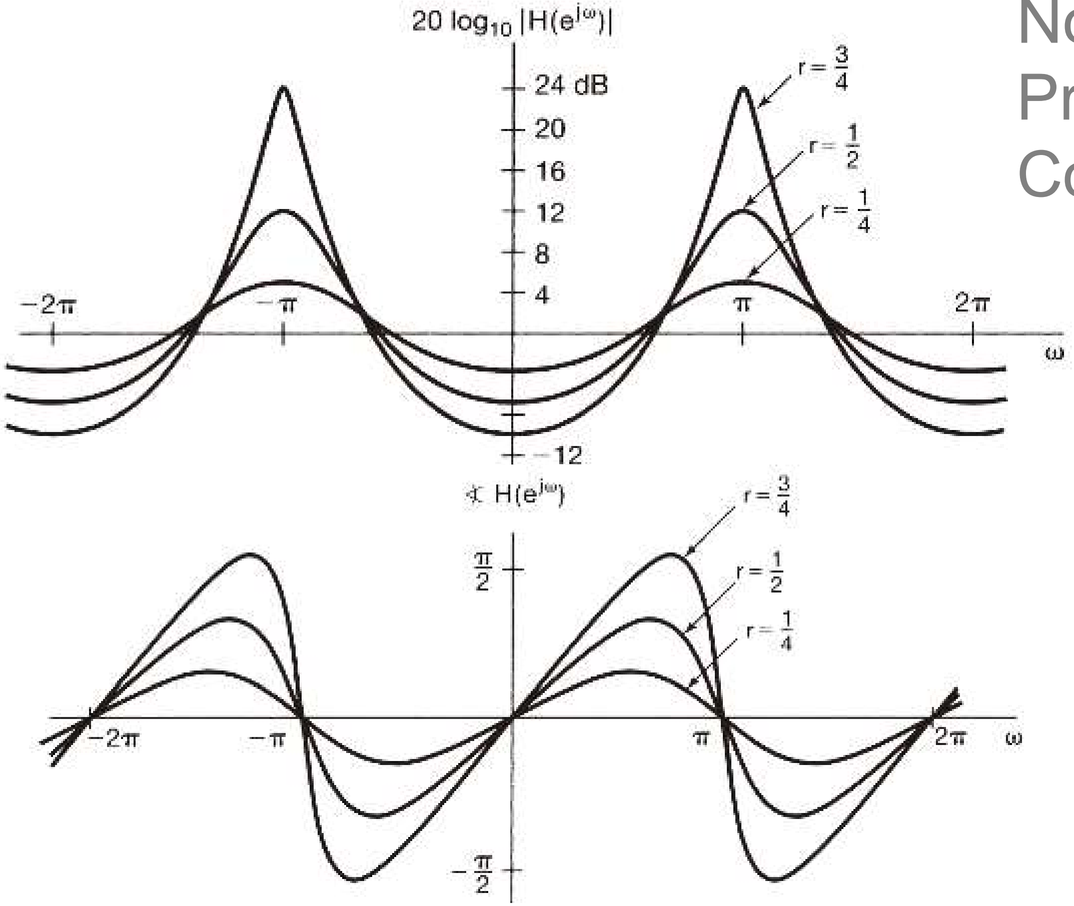
But when ζ decreases, the step response overshoots and rings a lot.
Again, this shows that there is a time vs frequency domain trade-off.

Active Filter



6.7.2 Examples of discrete-Time Nonrecursive Filters

- The issue for recursive discrete-time filter is the nonlinear phase



Nonrecursive filters
Pros: linear phase
Cons: more delay element

6.7.2 Examples of discrete-Time Nonrecursive Filters

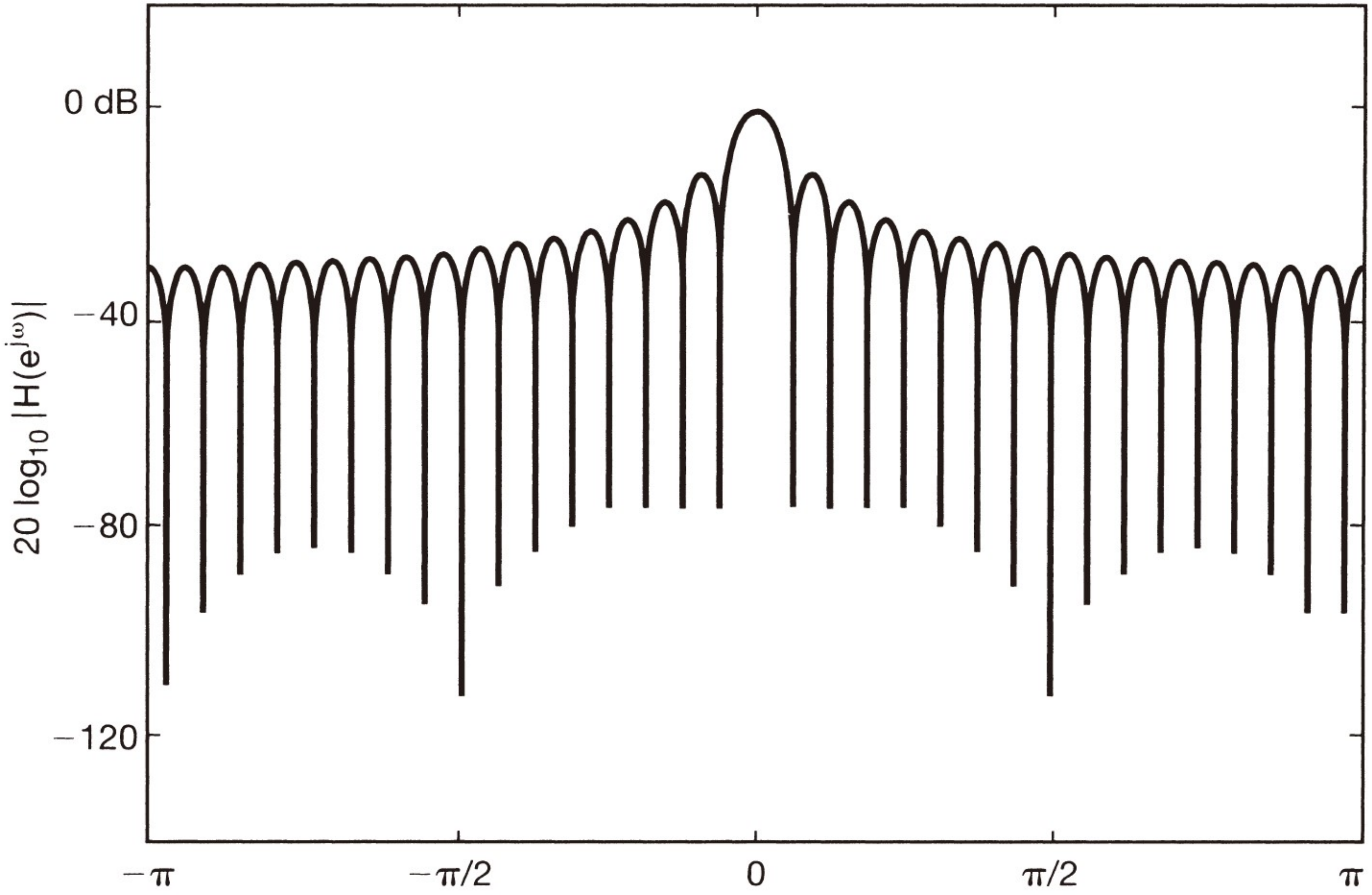
For this class of filters, the output is the average of the values of the input over a finite window:

$$y[n] = \frac{1}{N+M+1} \sum_{k=-N}^{M} x[n-k]. \qquad (6.78)$$

The corresponding impulse response is a rectangular pulse, and the frequency response is

$$H(e^{j\omega}) = \frac{1}{N+M+1} \boxed{e^{j\omega[(N-M)/2]}} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}. \qquad (6.79)$$

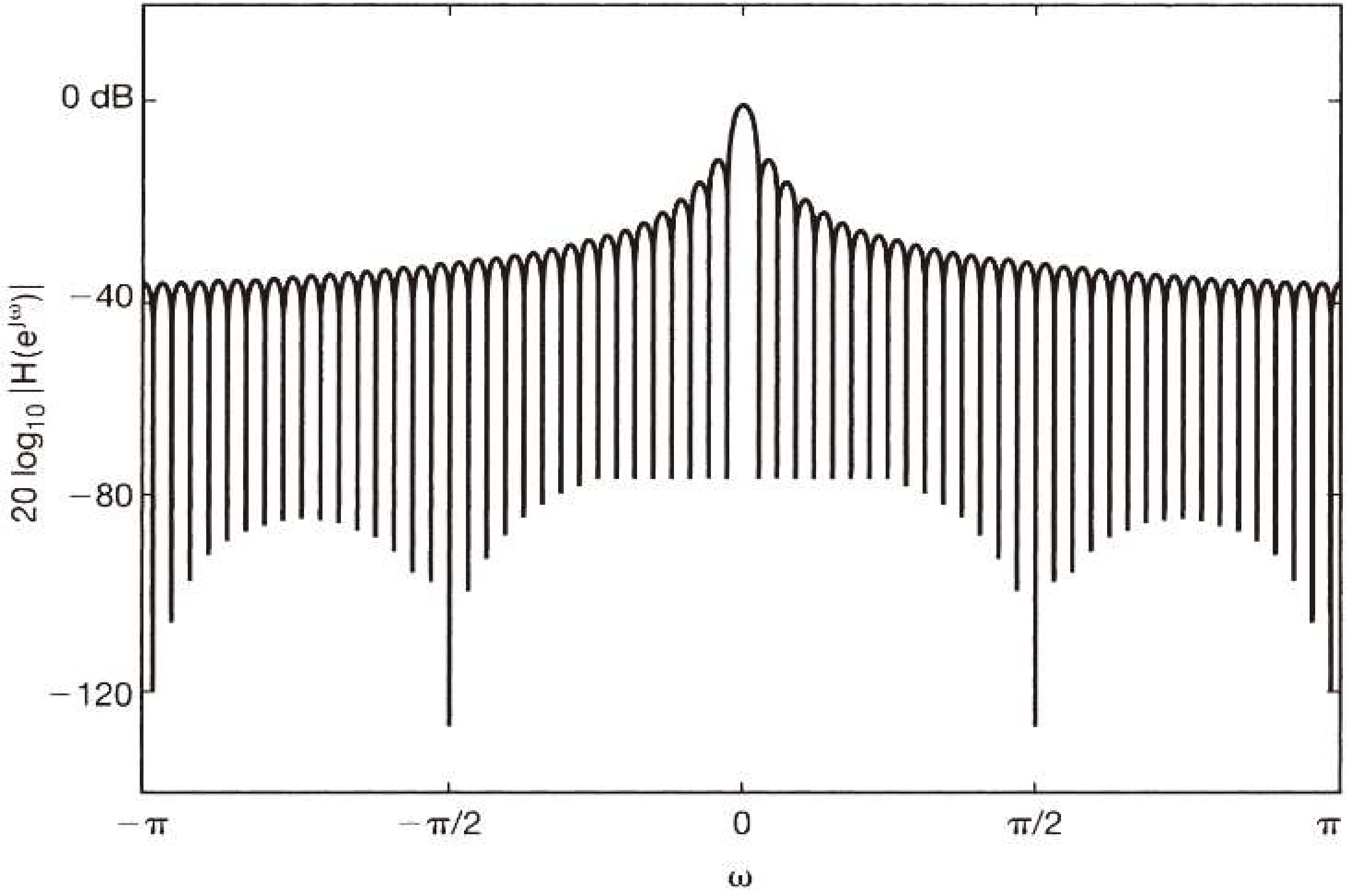
6.7.2 Examples of discrete-Time Nonrecursive Filters



(a)

Figure 6.35 Log-magnitude plots for the moving-average filter of eqs. (6.78) and (6.79) for (a) $M + N + 1 = 33$ and (b) $M + N + 1 = 65$.

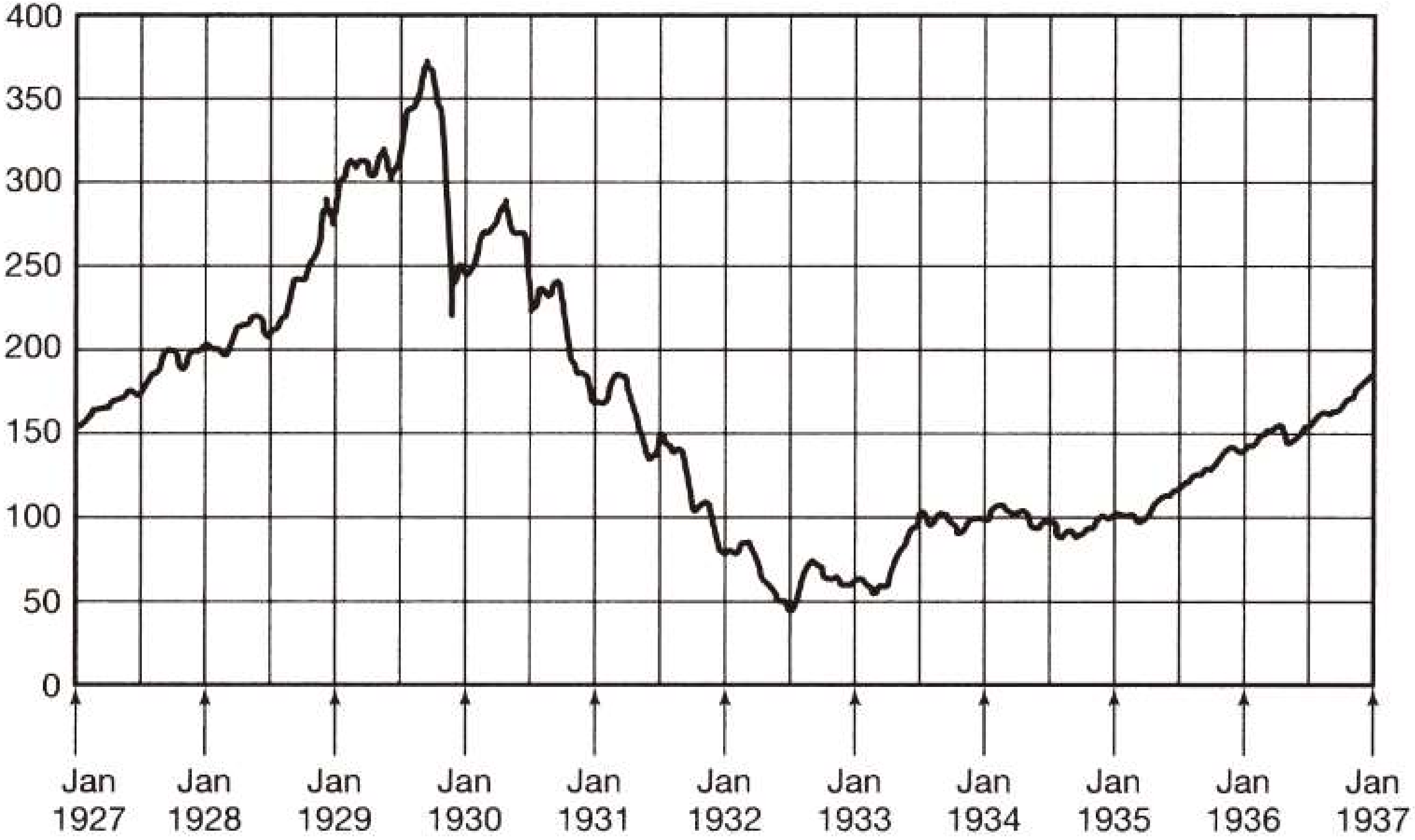
6.7.2 Examples of discrete-Time Nonrecursive Filters



(b)

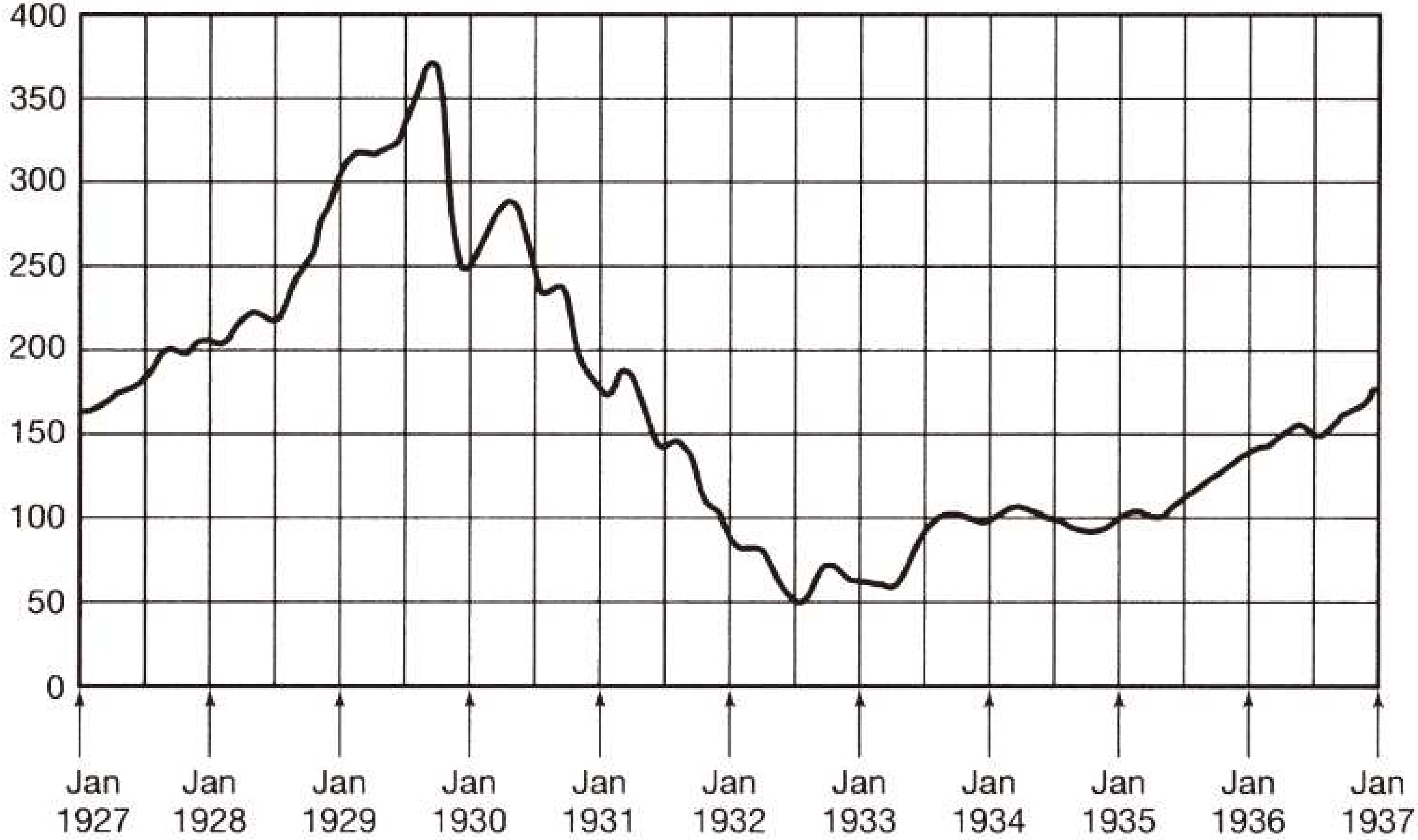
Figure 6.35 Log-magnitude plots for the moving-average filter of eqs. (6.78) and (6.79) for (a) $M + N + 1 = 33$ and (b) $M + N + 1 = 65$.

6.7.2 Moving average in Dow Jones index



(a)

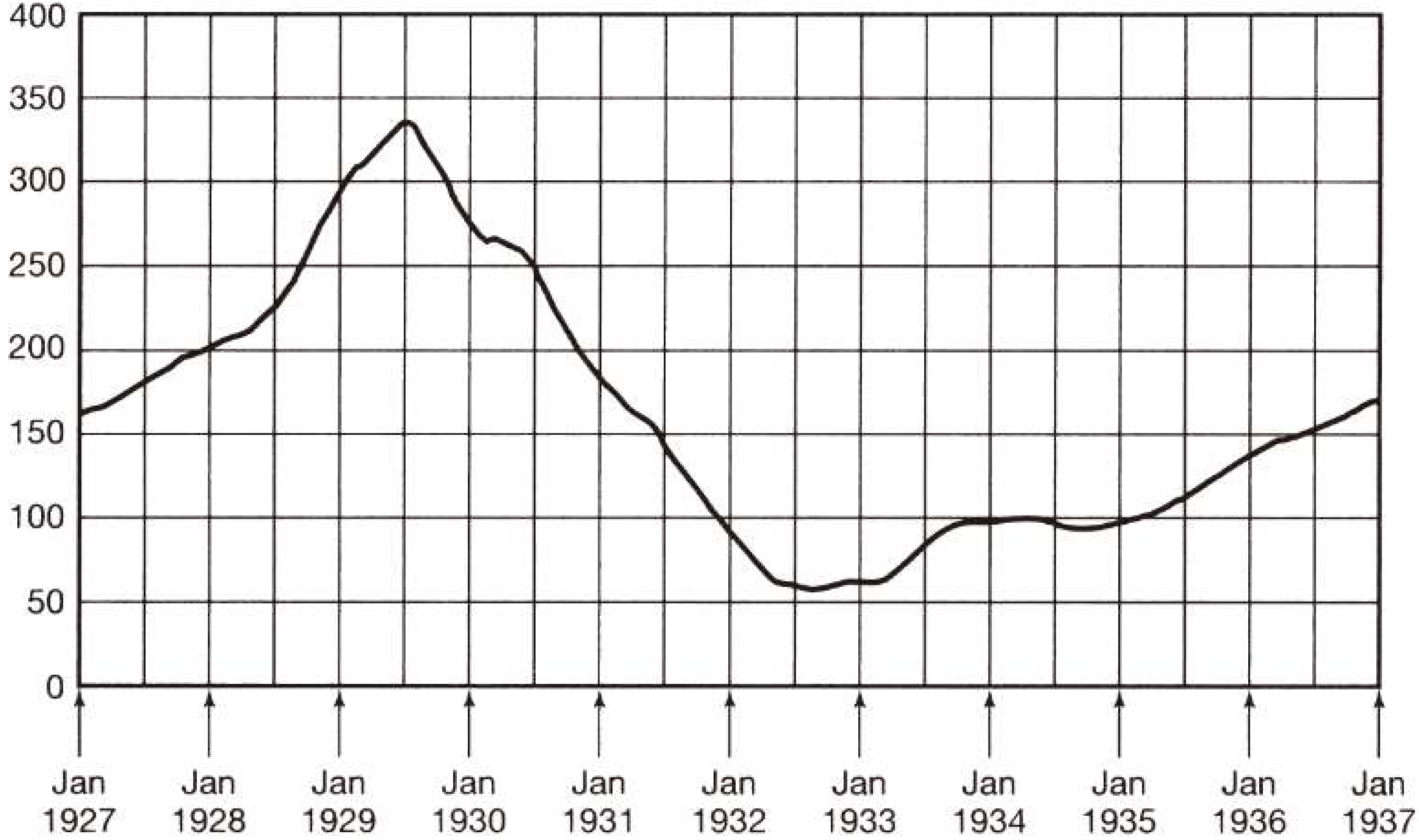
6.7.2 Moving average in Dow Jones index



(b)

51 days moving average

6.7.2 Moving average in Dow Jones index



(c)

100 days moving average

6.7.2 Examples of discrete-Time Nonrecursive Filters

The more general form of a discrete-time nonrecursive filter is

$$y[n]=\sum_{k=-N}^{M} b_k x[n-k], \qquad (6.80)$$

To illustrate how adjustment of the coefficients can influence the response of the filter, let us consider a filter of the form of eq. (6.80), with $N = M$ =16 and the filter coefficients chosen to be

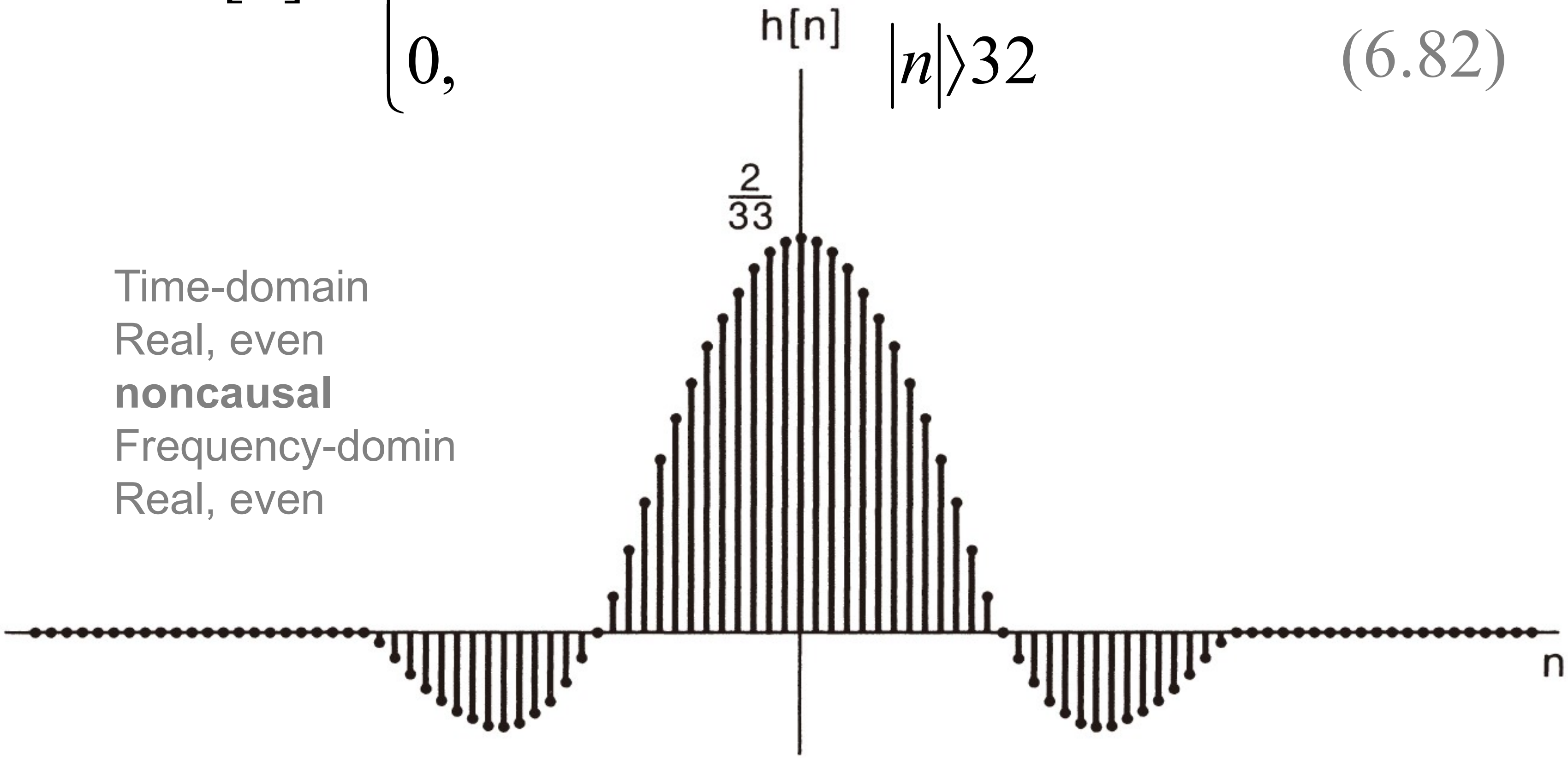
$$b_k=\begin{cases}\dfrac{\sin(2\pi k/33)}{\pi k}, & |k|\le 32\\ 0, & |k|\rangle 32\end{cases}. \qquad (6.81)$$

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k], \qquad b_k = \begin{cases} \dfrac{\sin(2\pi k/33)}{\pi k}, & |k| \le 32 \\ 0, & |k| \rangle 32 \end{cases}.$$

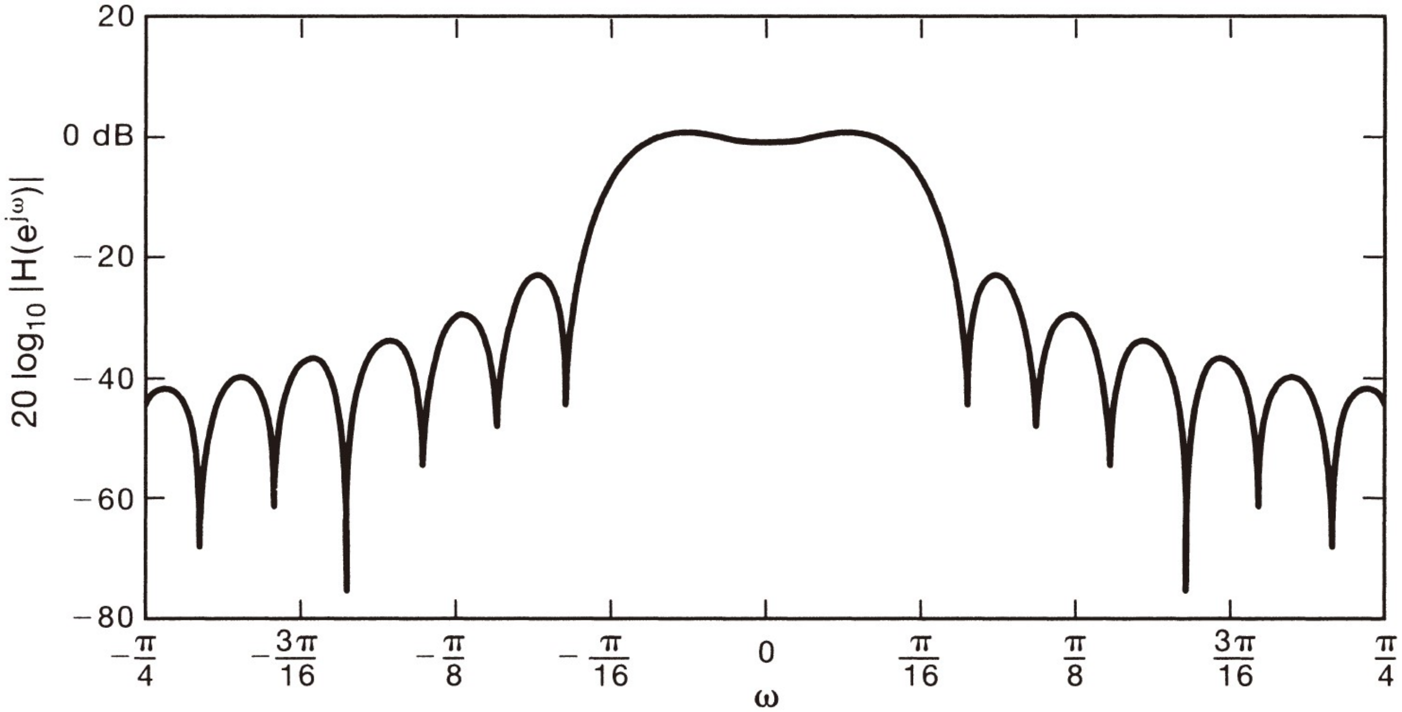
The impulse response of the filter is

$$h[n] = \begin{cases} \dfrac{\sin(2\pi n/33)}{\pi n}, & |n| \le 32 \\ 0, & |n| \rangle 32 \end{cases}. \qquad (6.82)$$

Time-domain
Real, even
noncausal
Frequency-domin
Real, even



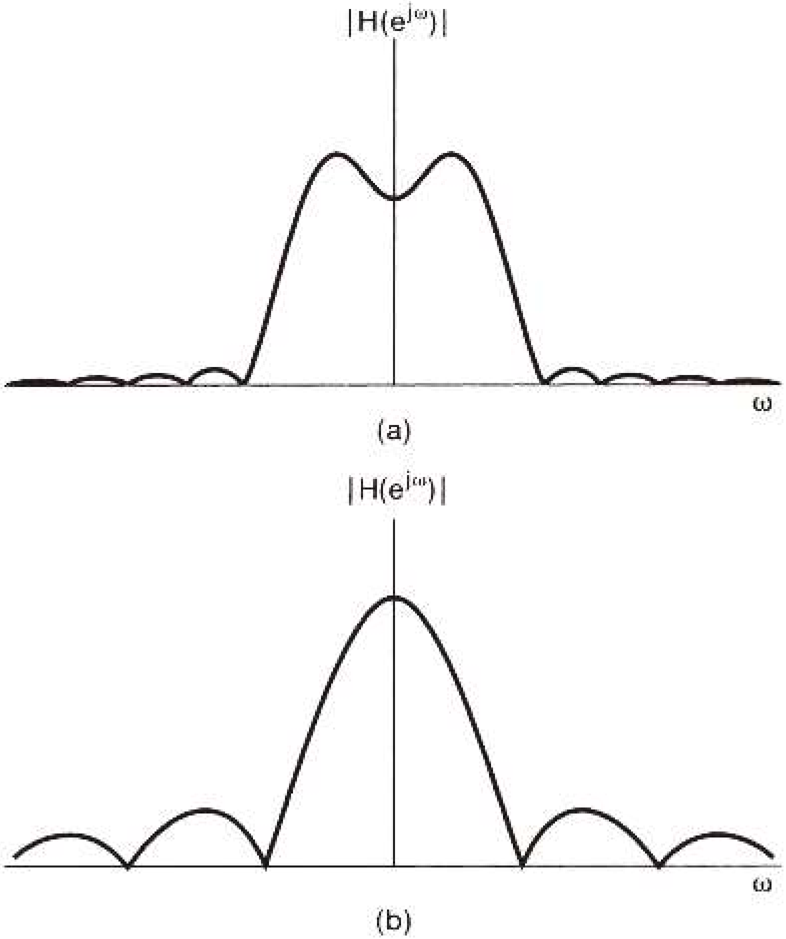
6.7.2 Examples of discrete-Time Nonrecursive Filters



(b)

Figure 6.37 (a) Impulse response for the nonrecursive filter of eq. (6.82); (b) log magnitude of the frequency response of the filter.

6.7.2 Examples of discrete-Time Nonrecursive Filters



Truncated Sinc

Moving Average

Figure 6.38 Comparison, on a linear amplitude scale, of the frequency responses of (a) Figure 6.37 and (b) Figure 6.35.

6.7.2 Examples of discrete-Time Nonrecursive Filters

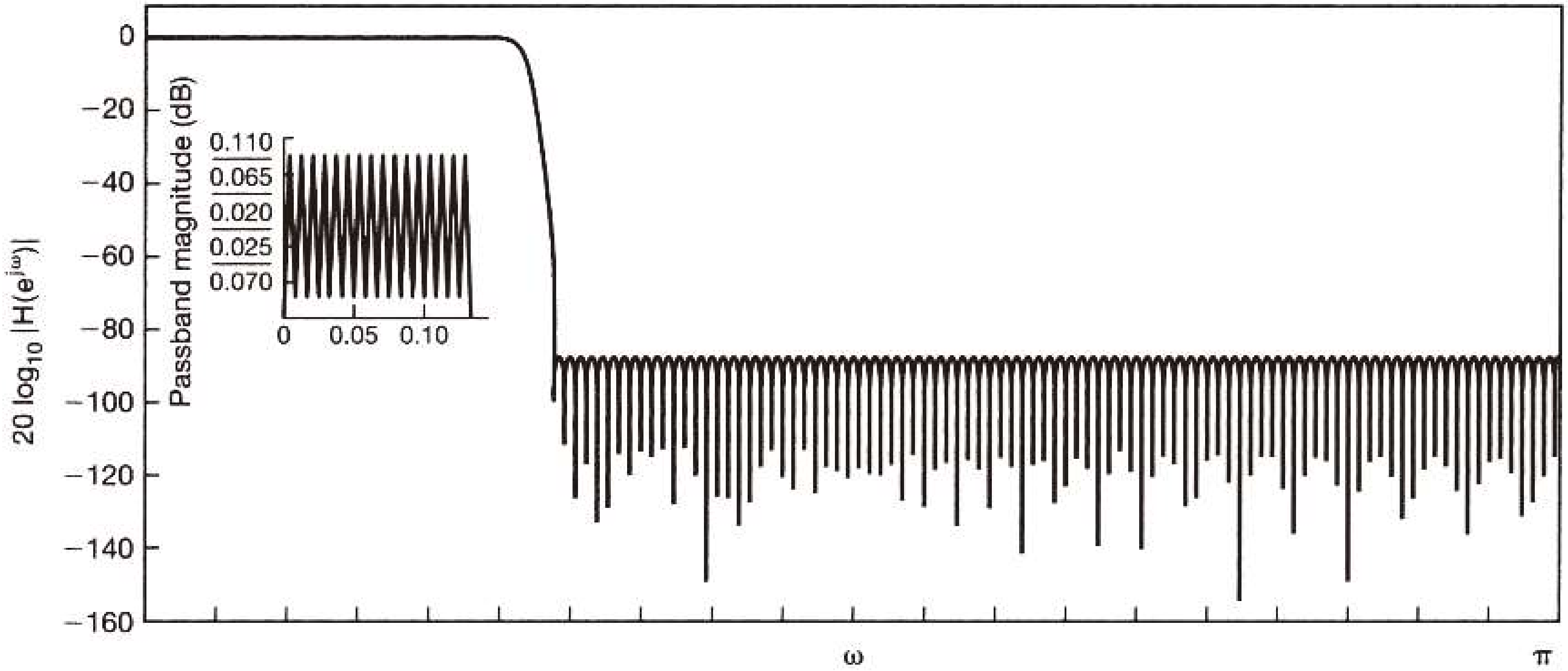


Figure 6.39 Lowpass nonrecursive filter with 251 coefficients designed to obtain the sharpest possible cutoff.

6.7.2 Examples of discrete-Time Nonrecursive Filters

Now define the nonrecursive LTI system resulting from a simple N-step delay of $h[n]$, i.e.,

$$h_1[n] = h[n-N], \tag{6.83}$$

then $h_1[n] = 0$ for all $n < 0$, so that this LTI system is causal. Furthermore, from the time-shift property for discrete-time Fourier transforms, we see that the frequency response of the system is

$$H_1\left(e^{j\omega}\right) = H\left(e^{j\omega}\right)\boxed{e^{-j\omega N}}. \tag{6.84}$$

6.8 Summary

- Magnitude-Phase Representation
- Bode Plot (log Magnitude and log frequency)
- Phase Distortion & Group Delay
- Time-domain Properties
- Frequency-domain Properties
- 1st and 2nd –order Continuous-time System
- 1st and 2nd –order Discrete-time System
- Higher-order System from 1st and 2nd-order Systems
- Automobile Suspension Systems
- Nonrecursive Discrete-time System