Chapter 5
The Discrete-Time Fourier Transform

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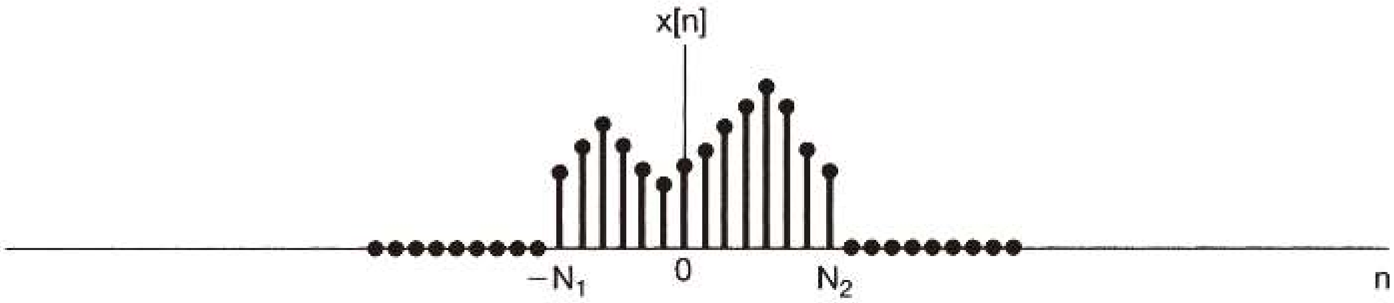
5.0 Introduction

In the chapter, we take advantage of the similarities between continuous-time and discrete-time Fourier analysis by following a strategy essentially identical to that used in Chapter 4.

在本章後段將利用連續時間與離散時間傅立葉分析的相似特性，來進行與第4章相同的分析策略。

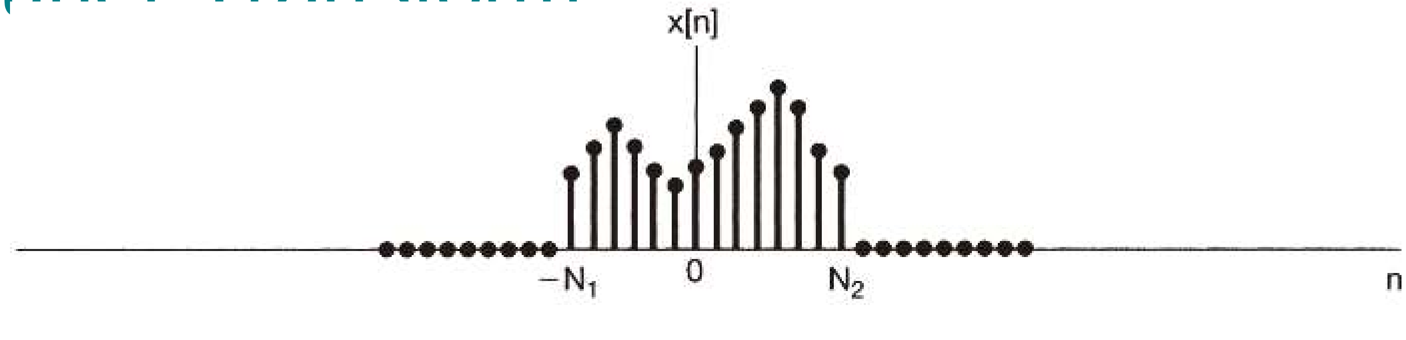
5.1.1 Development of the Discrete-Time Fourier Transform

Consider a general sequence *x[n]* that is of **finite** duration. That is, for some integers N_1 and N_2, *x[n]* = 0 outside the range $-N_1 \le n \le N_2$. A signal of this type is illustrated in Figure 5.1(a).



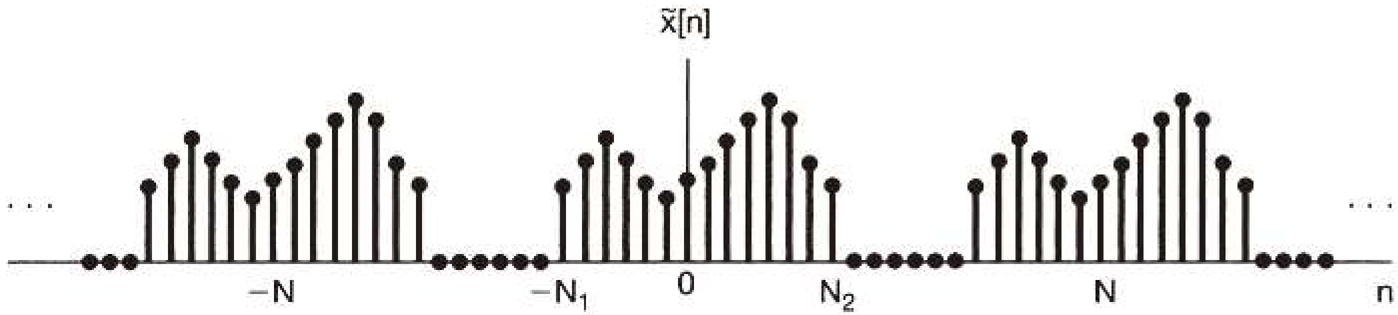
(a)

5.1.1 Development of the Discrete-Time Fourier Transform



(a)

From this aperiodic signal, we can construct a periodic signal $\tilde{x}[n]$ for which *x[t]* is one period, as indicated in Figure 5.1(b).



(b)

5.1.1 Development of the Discrete-Time Fourier Transform

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} \tilde{x}[n] e^{-jk(2\pi/N)n} \qquad (5.2)$$

Since $x[n] = \tilde{x}[n]$ over a period that includes the interval $-N_1 \le n \le N_2$, it is convenient to choose the interval of summation in eq. (5.2) to include this interval, to that $\tilde{x}[n]$ can be replaced by *x[n]* in the summation. Therefore,

$$a_k = \frac{1}{N} \sum_{n=-N_1}^{N_2} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N} \sum_{n=-\infty}^{+\infty} x[n] e^{-jk(2\pi/N)n}, \qquad (5.3)$$

5.1.1 Development of the Discrete-Time Fourier Transform

$$a_k = \frac{1}{N}\sum_{n=-\infty}^{+\infty} x[n]e^{-jk\omega_0 n} \quad (5.3)$$

Where in the second equality in eq. (5.3) we have used the fact that *x[n]* is zero outside the interval $-N_1 \le n \le N_2$. Defining the function

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \quad (5.4)$$

we see that the coefficients a_k are proportional to samples of $X(e^{j\omega})$, i.e.,

$$a_k = \frac{1}{N}X(e^{jk\omega_0}), \quad (5.5)$$

$$X(e^{j\omega+2\pi}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j(\omega+2\pi)n} = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}e^{-j2\pi n} = X(e^{j\omega})$$

5.1.1 Development of the Discrete-Time Fourier Transform

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} a_k e^{jk\omega_0 n} \quad (5.1) \qquad a_k = \frac{1}{N} X(e^{jk\omega_0}) \quad (5.5)$$

Where $\omega_0 = 2\pi / N$ is the spacing of the samples in the frequency domain. Combining eqs. (5.1) and (5.5) yields

$$\tilde{x}[n] = \sum_{k=\langle N \rangle} \frac{1}{N} X(e^{jk\omega_0}) e^{jk\omega_0 n}. \tag{5.6}$$

Since $\omega_0 = 2\pi / N$, or equivalently, $1/N = \omega_0 / 2\pi$ eq. (5.6) can be rewritten as

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0. \tag{5.7}$$

5.1.1 Development of the Discrete-Time Fourier Transform

$$\tilde{x}[n] = \frac{1}{2\pi} \sum_{k=\langle N \rangle} X(e^{jk\omega_0}) e^{jk\omega_0 n} \omega_0 .$$

As $N \to \infty,\quad \tilde{x}[n] = x[n]$, and eq. (5.7) becomes

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega,$$

當 $N \to \infty, \tilde{x}[n] \to x[n]$，可得*x[n]*與$X(e^{j\omega})$ 關係式。

Since $X(j^{j\omega})e^{j\omega n}$ is periodic with period 2*π*, the interval of integration can be taken as any interval of length 2*π*.

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega, \quad (5.9)$$

spectrum

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}.$$

5.1.1 Development of the Discrete-Time Fourier Transform

As our derivation indicates, the discrete-time Fourier transform shares many similarities with the continuous-time case. The major differences between the two are **the periodicity of the discrete-time transform** $X(e^{j\omega})$ and the **finite interval of integration (2*π*)** in the synthesis equation.

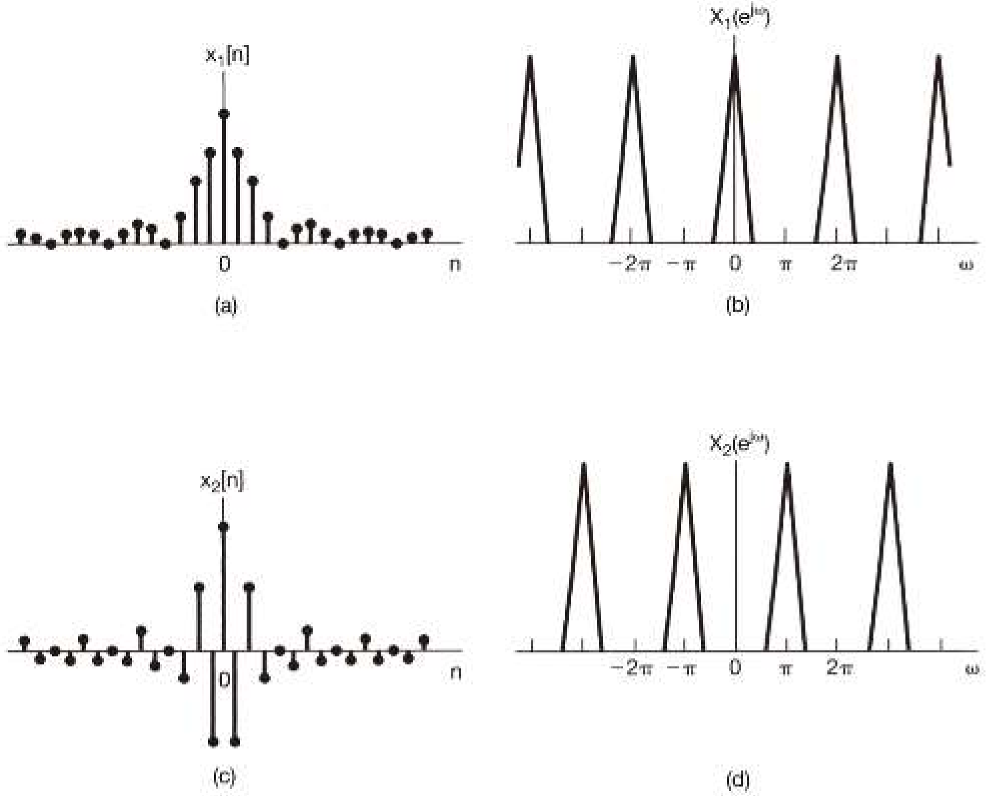
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.(5.9)$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega(4.8)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt(4.9)$$

5.1.1 Development of the Discrete-Time Fourier Transform



Lower frequency

Higher frequency

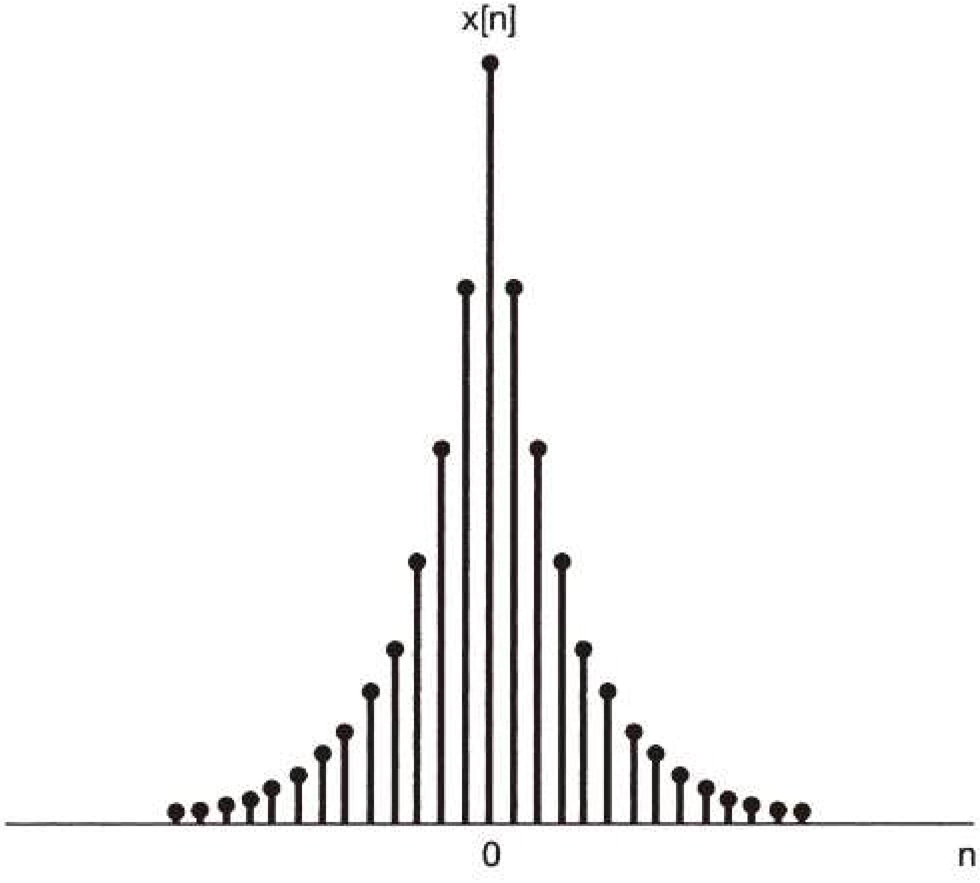
Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

Example 5.2

Let

$$x[n] = a^{|n|}, \qquad |a| < 1.$$

This signal is sketched for $0 < a < 1$ in Figure 5.5(a). Its Fourier Transform is obtained from eq. (5.9):



(a)

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi} \int_{2\pi} X(e^{j\omega}) e^{j\omega n} d\omega,$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}.$$

Example 5.2

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\,\frac{1-r^n}{1-r},$$

$$X(e^{j\omega}) = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n + \sum_{n=1}^{\infty} (ae^{j\omega})^n$$

$$= \frac{1}{1-ae^{-j\omega}} + \frac{ae^{j\omega}}{1-ae^{j\omega}} = \frac{1-a^2}{1-2a\cos(\omega)+a^2}$$

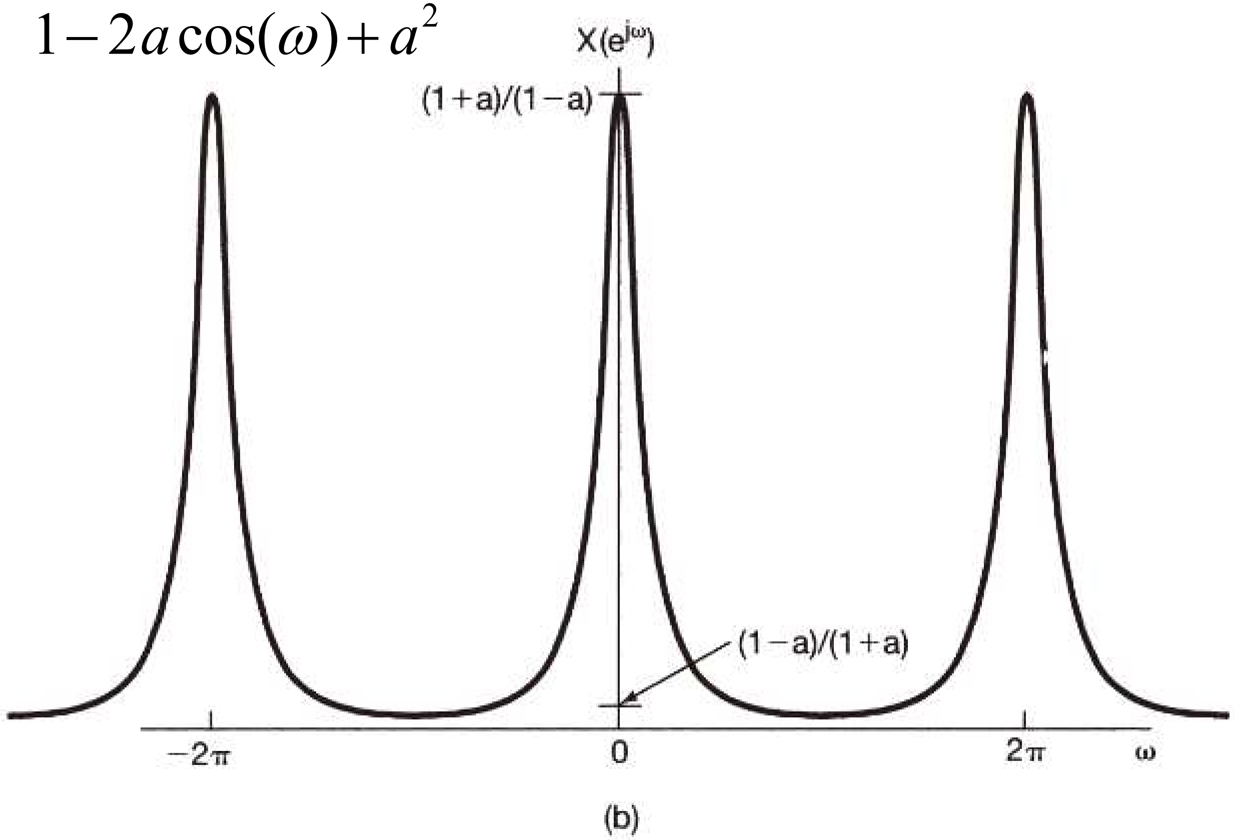


Figure 5.5 (a) Signal $x[n] = a^{|n|}$ of Example 5.2 and (b) its Fourier transform $(0 < a < 1)$.

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.(5.9)$$

Specifically, eq. (5.9) will converge either if *x[n]* is absolutely summable, that ,is

$$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty, \qquad (5.13)$$

若*x[n]*為絕對可加，則其傅立葉轉換存在。

or if the sequence has finite energy, that is,

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty. \qquad (5.14)$$

若*x[n]*具有有限能量，則其傅立葉轉換存在。

5.1.3 Convergence Issues Associated with the Discrete-Time Fourier Transform

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

In particular, if we approximate an aperiodic signal *x[n]* by an integral of complex exponentials with frequencies taken over the interval $|\omega| \le W$, i.e.,

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} X(e^{j\omega})e^{j\omega n}d\omega. \qquad (5.15)$$

when *W=π*，*x[n] equals* $\hat{x}[n]$ *in* Eq. 5.15

Example 5.4

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.(5.9)$$

Let *x[n]* be the unit impulse; that is,

$$x[n] = \delta[n].$$

In this case the analysis equation (5.9) is easily evaluated, yielding

$$X\left(e^{j\omega}\right) = 1.$$

Example 5.4

In other words, just as in continuous time, the unit impulse has Fourier transform representation consisting of equal contributions at all frequencies. If we then apply eq.(5.15) to this example, we obtain

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n}\,d\omega = \frac{1}{2\pi}\frac{1}{jn}e^{j\omega n}\Big|_{-W}^{W} = \frac{e^{j\omega W} - e^{-j\omega W}}{2\pi jn}$$

$$= \frac{\sin Wn}{\pi n} = \frac{W}{\pi}\frac{\sin Wn}{Wn}. \qquad (5.16)$$

$$\hat{x}[n] = \frac{W}{\pi}\frac{\sin Wn}{Wn}.$$

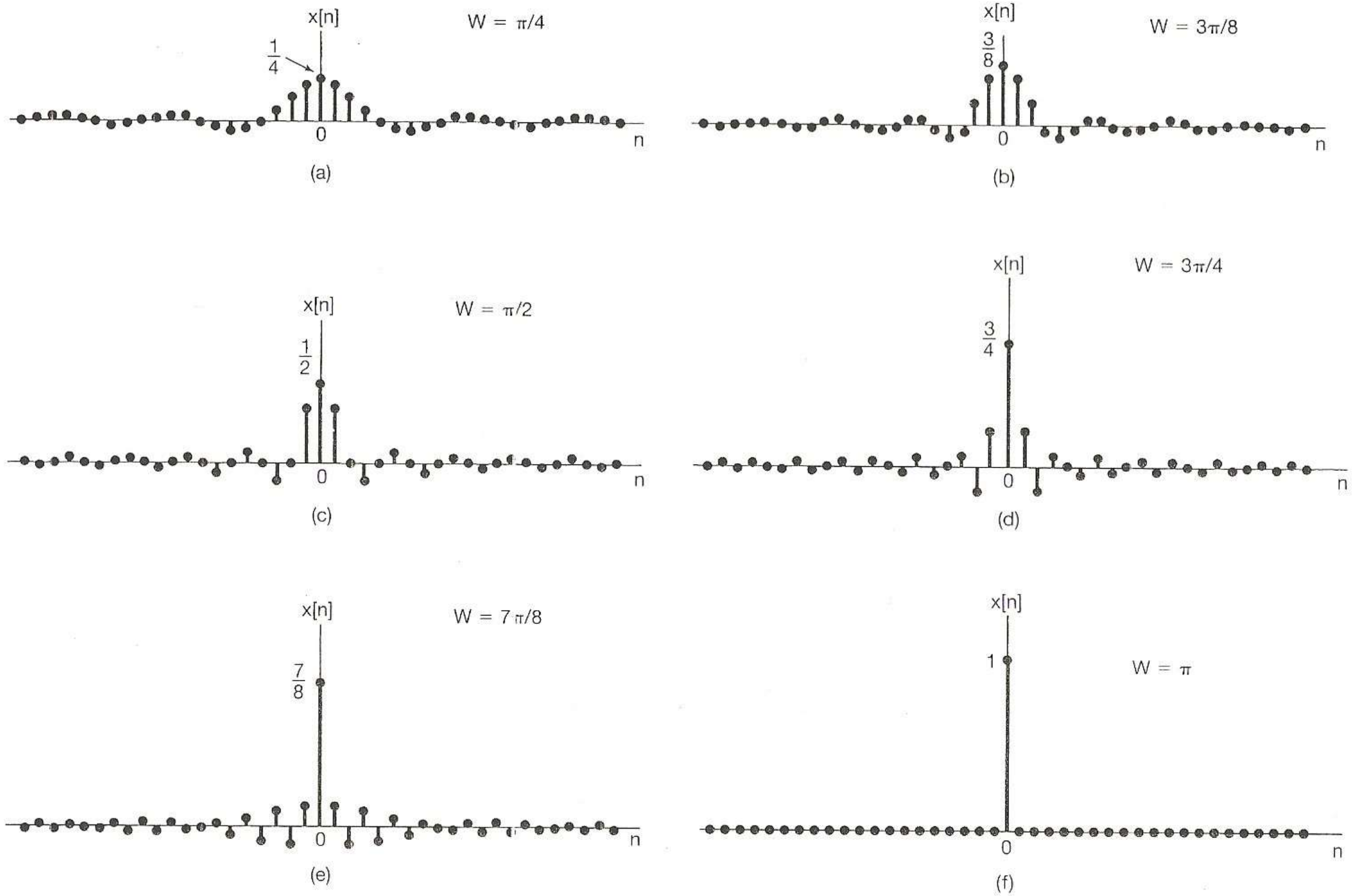


Figure 5.7 Approximation to the unit sample obtained as in eq. (5.16) using complex exponentials with frequencies $|\omega| \leq W$: (a) $W = \pi/4$; (b) $W = 3\pi/8$; (c) $W = \pi/2$; (d) $W = 3\pi/4$; (e) $W = 7\pi/8$; (f) $W = \pi$. Note that for $W = \pi$, $\hat{x}[n] = \delta[n]$.

5.2 The Fourier Transform for Periodic Signals

As in the continuous-time case, discrete-time **periodic** signals can be incorporated within the framework of the discrete-time Fourier transform by interpreting the transform of a periodic signal as an **impulse train** in the frequency domain.

To derive the form of this representation, consider the signal

$$x[n] = e^{j\omega_0 n}. \qquad (5.17)$$

In continuous time, the FT of $e^{j\omega 0t}$ is an impulse at $\omega=\omega_0$. However, DTFT must be periodic with period 2π

5.2 The Fourier Transform for Periodic Signals

The Fourier transform of $x[n] = e^{j\omega_0 n}$ is the impulse train

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - (\omega_0 + 2\pi l)), \qquad (5.18)$$

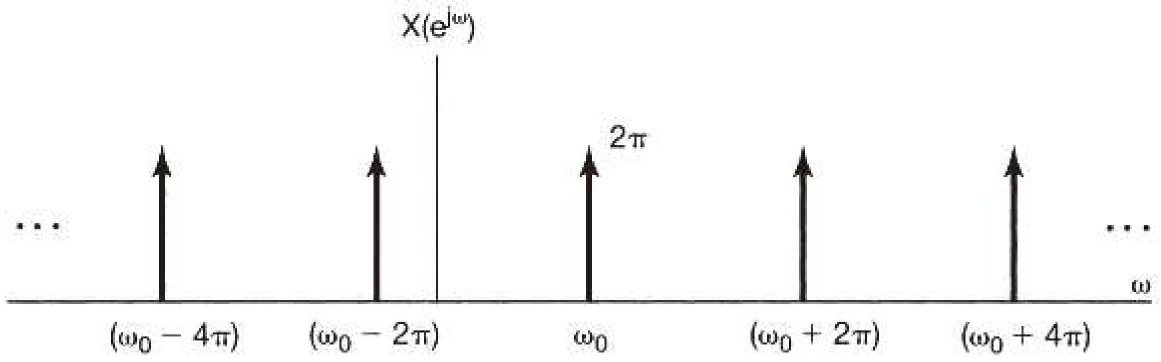


Figure 5.8 Fourier transform of $x[n] = e^{j\omega_0 n}$.

5.2 The Fourier Transform for Periodic Signals

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - (\omega_0 + 2\pi l)),$$

In order to check the validity of this expression, we must evaluate its inverse transform. Substituting eq. (5.18) into the synthesis equation (5.8), we find that

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)e^{j\omega n}d\omega.$$

5.2 The Fourier Transform for Periodic Signals

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - (\omega_0 + 2\pi l)),$$

If the interval of integration chosen includes the impulse located at $\omega_0 + 2\pi r$, then

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega = e^{j(\omega_0 + 2\pi r)n} = e^{j\omega_0 n}.$$

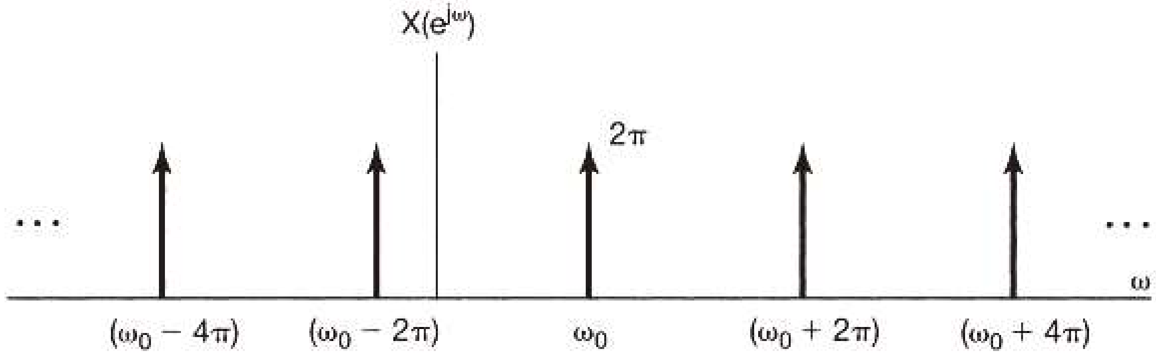


Figure 5.8 Fourier transform of $x[n] = e^{j\omega_0 n}$.

5.2The Fourier Transform for Periodic Signals

$$x[n]=e^{j\omega_0 n}\leftrightarrow X(e^{j\omega})=\sum_{l=-\infty}^{+\infty}2\pi\delta(\omega-(\omega_0+2\pi l))$$

Now consider a periodic sequence *x[n]* with period *N* and with the Fourier series representation

$$x[n]=\sum_{k=\langle N\rangle}a_k e^{jk(2\pi/N)n}. \tag{5.19}$$

In this case, the Fourier transform is

$$X(e^{j\omega})=\sum_{k=-\infty}^{+\infty}2\pi a_k\delta\left(\omega-\frac{2\pi k}{N}\right), \tag{5.20}$$

so the FT of periodic signal can be directly constructed from FS coefficients

*x[n]*的傅立葉轉換(可由傅立葉係數直接建立)

5.2 The Fourier Transform for Periodic Signals

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

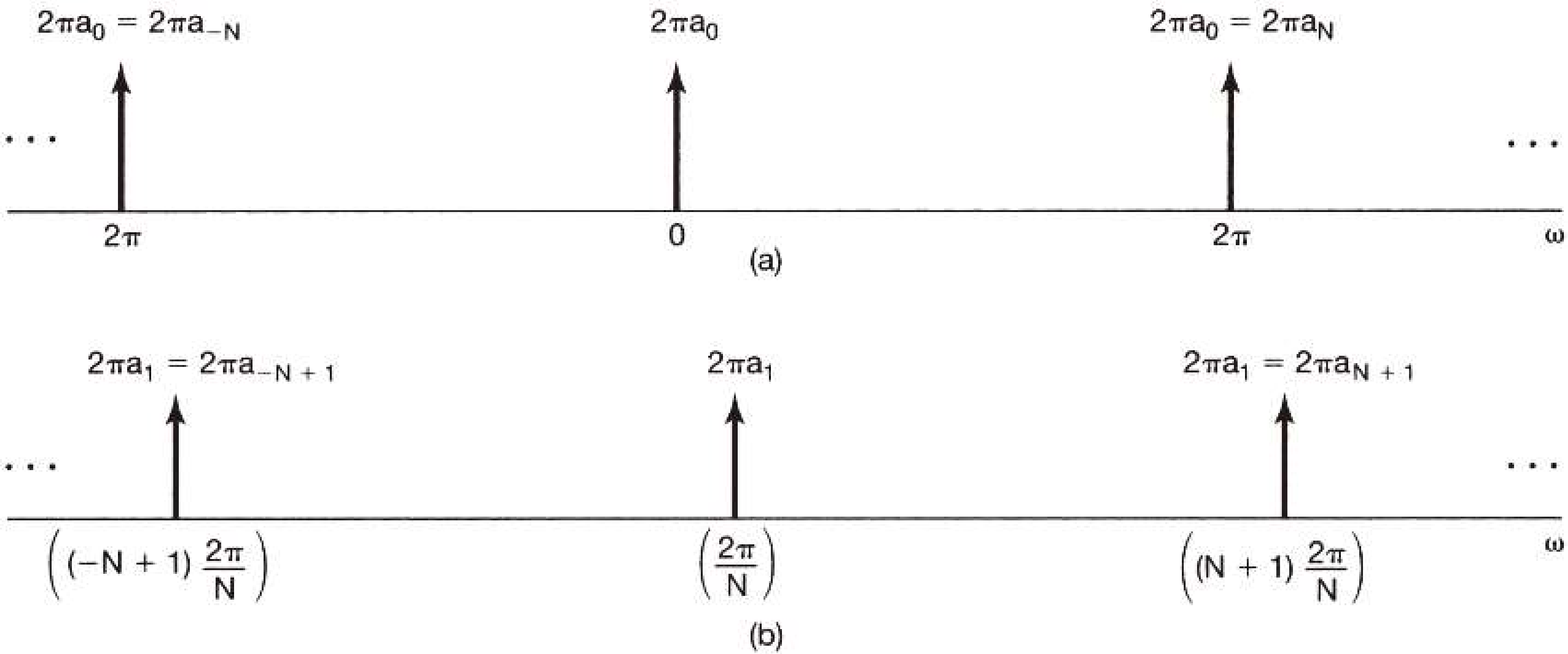
In particular, suppose that we choose the interval of summation in eq. (5.19) as *k* =0,1,…, *N*-1, so that

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \ldots + a_{N-1} e^{j(N-1)(2\pi/N)n}. \qquad (5.21)$$

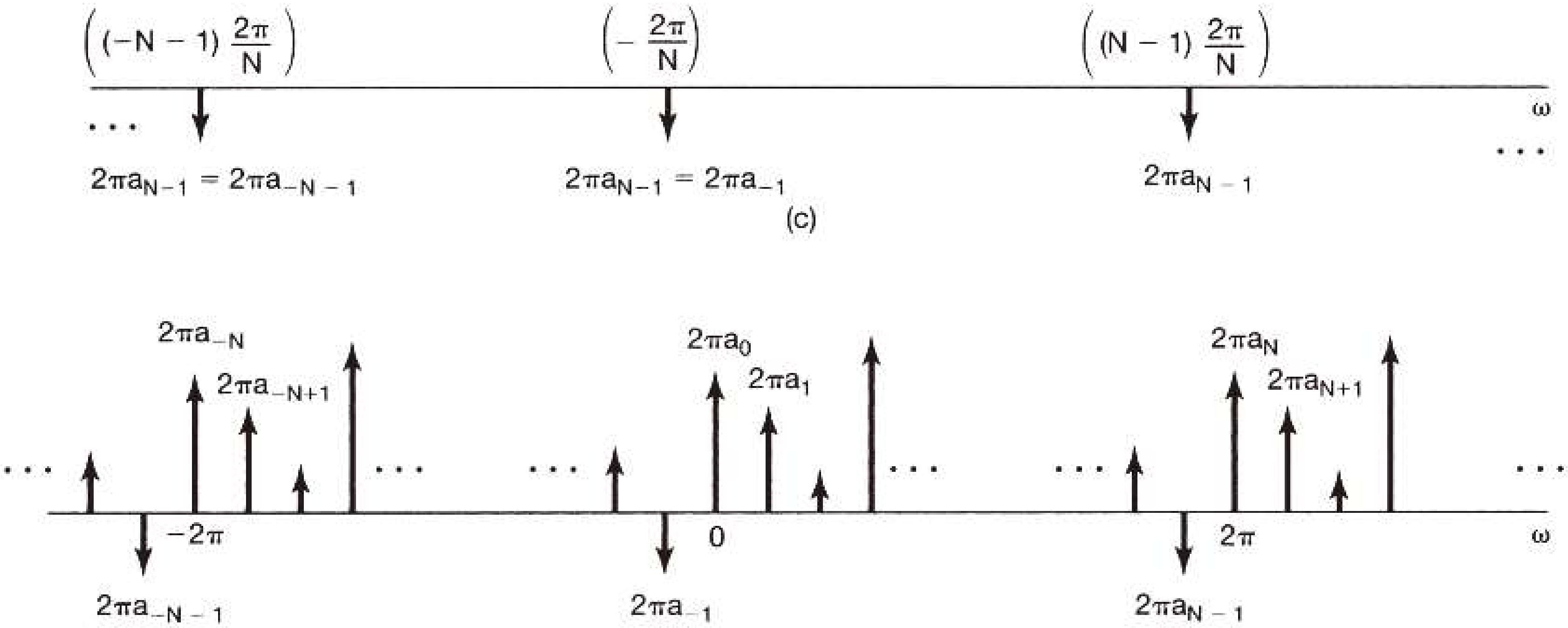
5.2 The Fourier Transform for Periodic Signals

$$a_k; \omega_0 = k(2\pi / N)$$

$$x[n] = e^{j\omega_0 n}. \quad X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - (\omega_0 + 2\pi l)),$$



5.2 The Fourier Transform for Periodic Signals



(c)

(d)

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{k=l\times N}^{l\times N+N-1} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

Example 5.5

$$x[n] = e^{j\omega_0 n}. \quad X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - (\omega_0 + 2\pi l)),$$

Consider the periodic signal

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \qquad with \qquad \omega_0 = \frac{2\pi}{5}. \quad (5.22)$$

From eq. (5.18), we can immediately write

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} \pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right). \quad (5.23)$$

Example 5.5

That is,

$$X(e^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), \qquad -\pi \le \omega < \pi,$$

And $X\left(e^{j\omega}\right)$ repeats periodically with period of 2π, as illustrated in Figure 5.10.

Example 5.5

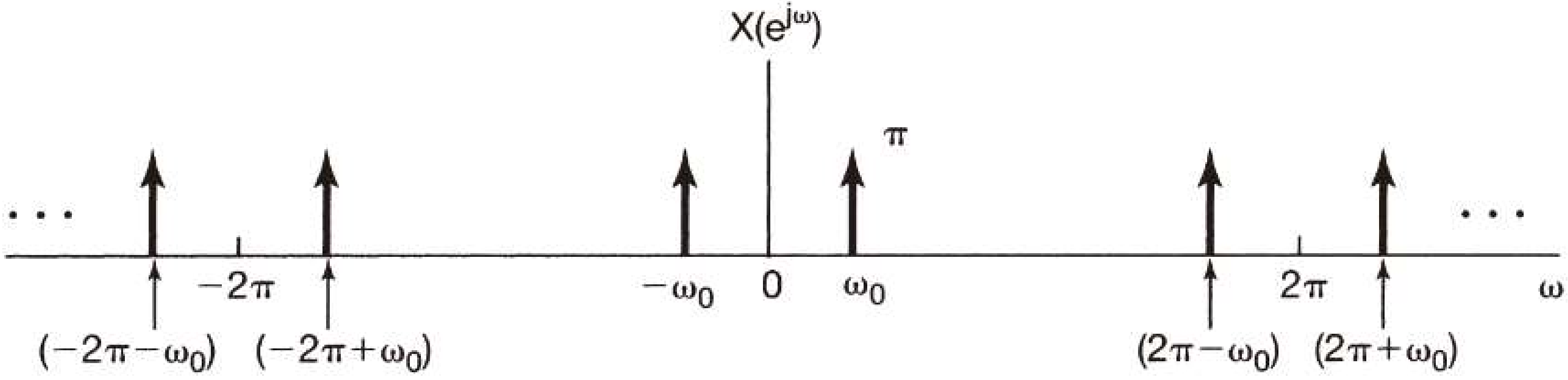


Figure 5.10 Discrete-time Fourier transform of $x[n] = \cos \omega_0 n$.

5.3 Properties of the Discrete-Time Fourier Transform

By comparing this table with Table 4.1, we can get a clear picture of some of the similarities and differences between continuous-time and discrete-time Fourier transform properties.

比較表4.1及5.1可瞭解連續時間與離散時間傅立葉轉換的一些異同點。

5.3 Properties of the Discrete-Time Fourier Transform

In the following discussions, it will be convenient to adopt notation similar to that used in Section 4.3 to indicate the pairing of a signal and its transform. That is,

傅立葉轉換

$$X(e^{j\omega}) = F\{x[n]\},$$

反傅立葉轉換

$$x[n] = F^{-1}\{X(e^{j\omega})\},$$

訊號與其轉換關係符號

$$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega}).$$

5.3.1 Periodicity of the Discrete-Time Fourier Transform

As we discussed in Section 5.1, the discrete-time Fourier transform is always periodic in ω with period 2π; i.e.,

$$X(e^{j(\omega+2\pi)}) = X(e^{j\omega}). \qquad (5.28)$$

$X(e^{j\omega})$必為週期函數，周期為2π。

This is in contrast to CTFT

5.3.2 Linearity of the Fourier Transform

If

$$x_1[n] \xleftrightarrow{F} X_1(e^{j\omega})$$

And

$$x_2[n] \xleftrightarrow{F} X_2(e^{j\omega}),$$

Then

$$ax_1[n] + bx_2[n] \xleftrightarrow{F} aX_1(e^{j\omega}) + bX_2(e^{j\omega}). \quad (5.29)$$

線性性質(重疊性質)

5.3.3 Time Shifting and Frequency Shifting

If $$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega}),$$

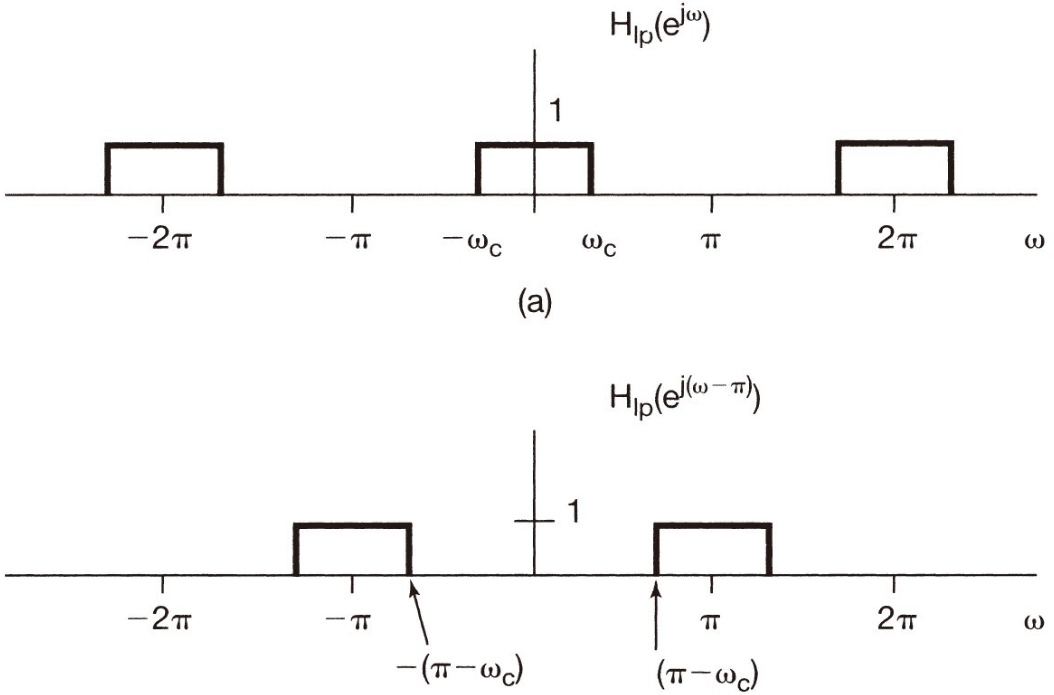
Then $$x[n-n_0] \overset{F}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega}) \qquad (5.30)$$

And $$e^{j\omega_0 n} x[n] \overset{F}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}). \qquad (5.31)$$

As a consequence of periodicity in frequency domain, there exist a special relation between ideal low-pass and high-pass filters.

Example 5.7

In Figure 5.12(a) we have depicted the frequency response $H_{1p}(e^{j\omega})$ of a lowpass filter with cutooff frequency ω_c , while in Figure 5.12(b) we have displayed $H_{1p}(e^{j(\omega-\pi)})$ -- that is, the frequency response $H_{1p}(e^{j\omega})$ shifted by one-half period (by *π*).



Since high frequencies in discrete time are concentrated near *π* (and other odd multiples of *π*), the filter in Figure 5.12(b) is an ideal highpass filter with cutoff frequency $\pi-\omega_c$. That is,

$$H_{hp}(e^{j\omega})=H_{1p}(e^{j(\omega-\pi)}).$$

Example 5.7

$$e^{j\omega_0 n}x[n] \stackrel{F}{\longleftrightarrow} X(e^{j(\omega-\omega_0)}).$$

As we can see from eq. (3.1222), and as we will discuss again in Section 5.4, the frequency response of an LTI system is the Fourier transform of the impulse response of the system. Thus, if $h_{lp}[n]$ and $h_{hp}[n]$ respectively denote the impulse response of the lowpass and highpass filter, and we know

$$H_{lp}(e^{j\omega}) \quad and \quad H_{hp}(e^{j\omega}) = H_{1p}(e^{j(\omega-\pi)})$$

the frequency-shifting property imply that

$$h_{hp}[n] = e^{j\pi n} h_{lp}[n] (5.33)$$

$$= (-1)^n \quad h_{lp}[n]. (5.34)$$

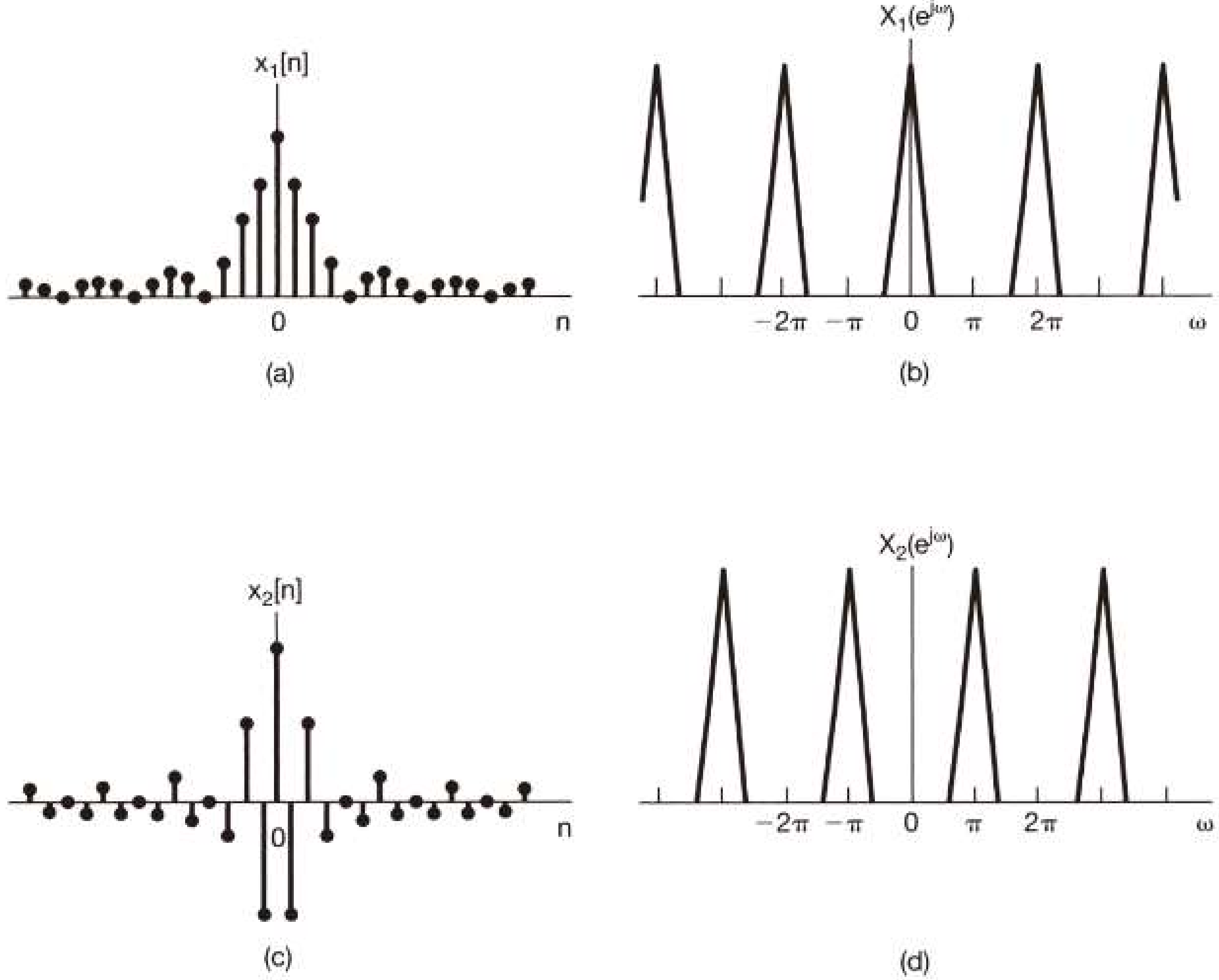


Figure 5.3 (a) Discrete-time signal $x_1[n]$. (b) Fourier transform of $x_1[n]$. Note that $X_1(e^{j\omega})$ is concentrated near $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$. (c) Discrete-time signal $x_2[n]$. (d) Fourier transform of $x_2[n]$. Note that $X_2(e^{j\omega})$ is concentrated near $\omega = \pm\pi, \pm 3\pi, \ldots$.

5.3.4 Conjugating and Conjugate Symmetry

If

$$x[n] \stackrel{F}{\longleftrightarrow} X(e^{j\omega}),$$

Then

$$x^*[n] \stackrel{F}{\longleftrightarrow} X^*(e^{-j\omega}). \qquad (5.35)$$

Also, if *x[n]* is real valued, its transform $X(e^{j\omega})$ is conjugate symmetric. That is

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad if \quad x[n] \quad is \quad real \qquad (5.36)$$

若*x[n]*為實值訊號，則$X(e^{j\omega})$具有共軛對稱性質。

5.3.4 Conjugating and Conjugate Symmetry

From this, it follows that $\Re e\left\{X(e^{j\omega})\right\}$ is an even function of ω and $\Im m\left\{X(e^{j\omega})\right\}$ is an odd function of ω. Similarly, the magnitude of $X(e^{j\omega})$ is an even function and the phase angle is an odd function. Furthermore,

$$\mathcal{E}v\left\{x[n]\right\} \overset{F}{\longleftrightarrow} \Re e\left\{X(e^{j\omega})\right\}$$

and

$$\mathcal{O}d\left\{x[n]\right\} \overset{F}{\longleftrightarrow} j\Im m\left\{X(e^{j\omega})\right\},$$

5.3.5 Differencing and Accumulation

From the linearity and time-shifting properties, the Fourier transform pair for the first-difference signal $x[n]-x[n-1]$ is given by

$$x[n]-x[n-1] \xleftrightarrow{F} (1-e^{-j\omega})X(e^{j\omega}). \qquad (5.37)$$

差分性質

Next, consider the signal

$$y[n]=\sum_{m=-\infty}^{n} x[m]. \qquad (5.38)$$

5.3.5 Differencing and Accumulation

$$y[n] = \sum_{m=-\infty}^{n} x[m].$$

Since $y[n] - y[n-1] = x[n]$, we might conclude that the transform of *y[n]* should be related to the transform of *x[n]* by division by $(1 - e^{-j\omega})$.
This is partly correct, but as with the continuous-time integration property given by eq. (4.32), there is more involved. The precise relationship is

$$\sum_{m=-\infty}^{n} x[m] \stackrel{F}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} X(e^{j\omega}) + \pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k).$$

(5.39)

Example 5.8 $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.(5.9)$

Let us derive the Fourier transform $X(e^{j\omega})$ of the unit step $x[n] = u[n]$ by making use of the accumulation property and the knowledge that

$$g[n] = \delta[n] \overset{F}{\longleftrightarrow} G(e^{j\omega}) = 1.$$

From Section 1.4.1 we know that the unit step is the running sum of the unit impulse. That is,

$$x[n] = \sum_{m=-\infty}^{n} g[m].$$

Example 5.8

Taking the Fourier transform of both sides and using accumulation yields

$$X(e^{j\omega}) = \frac{1}{(1-e^{-j\omega})} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

$$= \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k).$$

5.3.6 Time Reversal

Let *x[n]* be a signal with spectrum $X(e^{j\omega})$, and consider the transform $Y(e^{j\omega})$ of $y[n]=x[-n]$. From eq. (5.9),

$$Y(e^{j\omega})=\sum_{n=-\infty}^{+\infty}y[n]e^{-j\omega n}=\sum_{n=-\infty}^{+\infty}x[-n]e^{-j\omega n}. \qquad (5.40)$$

Substituting *m* = - *n* into eq.(5.40), we obtain

$$Y(e^{j\omega})=\sum_{m=-\infty}^{+\infty}x[m]e^{-j(-\omega)m}=X(e^{-j\omega}). \qquad (5.41)$$

That is,

$$x[-n]\overset{F}{\longleftrightarrow}X(e^{-j\omega}). \qquad (5.42)$$

時間倒轉性質

5.3.7 Time Expansion

In Section 4.3.5 we derived the continuous-time property

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right). \quad (5.43)$$

- x[an] is undefined when a not an integer
- X[an] (a>1 integer) doesn't only speed up but also sample

  There is a result that does closely parallel eq. (5.43), however. Let *k* be a positive integer, and define the signal

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k. \end{cases} \quad (5.44)$$

5.3.7 Time Expansion

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if n is a multiple of k} \\ 0, & otherwise \end{cases}$$

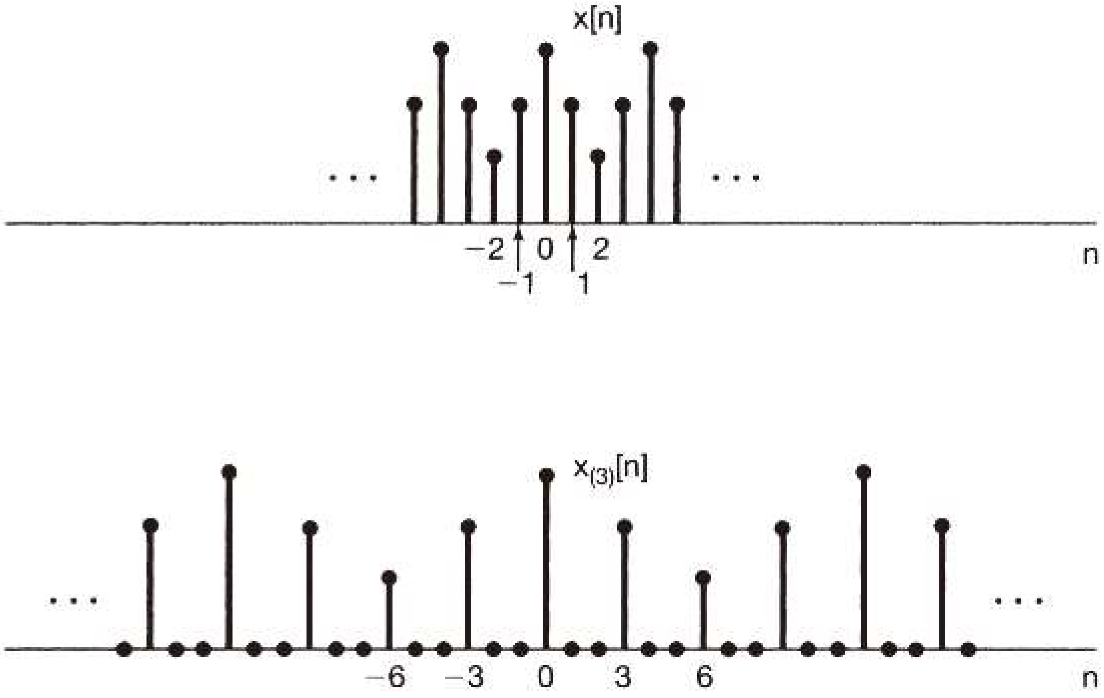


Figure 5.13 The signal $x_{(3)}[n]$ obtained from $x[n]$ by inserting two zeros between successive values of the original signal.

$$x_{(k)}[n] = \begin{cases} x[n/k] & \text{if n is a multiple of k} \\ 0, & otherwise \end{cases}$$

see that the Fourier transform of $x_{(k)}[n]$ is given by

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}.$$

Furthermore, since $x_{(k)}[rk] = x[r]$, we find that

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega}).$$

That is, $x_{(k)}[n] \overset{F}{\longleftrightarrow} X(e^{jk\omega}).$

When signal $x_{(k)}[n]$ is spread-out and slow down in time, its Fourier transform is compressed.

5.3.7 Time Expansion

$$x_{(k)}[n] \overset{F}{\longleftrightarrow} X(e^{jk\omega}).$$

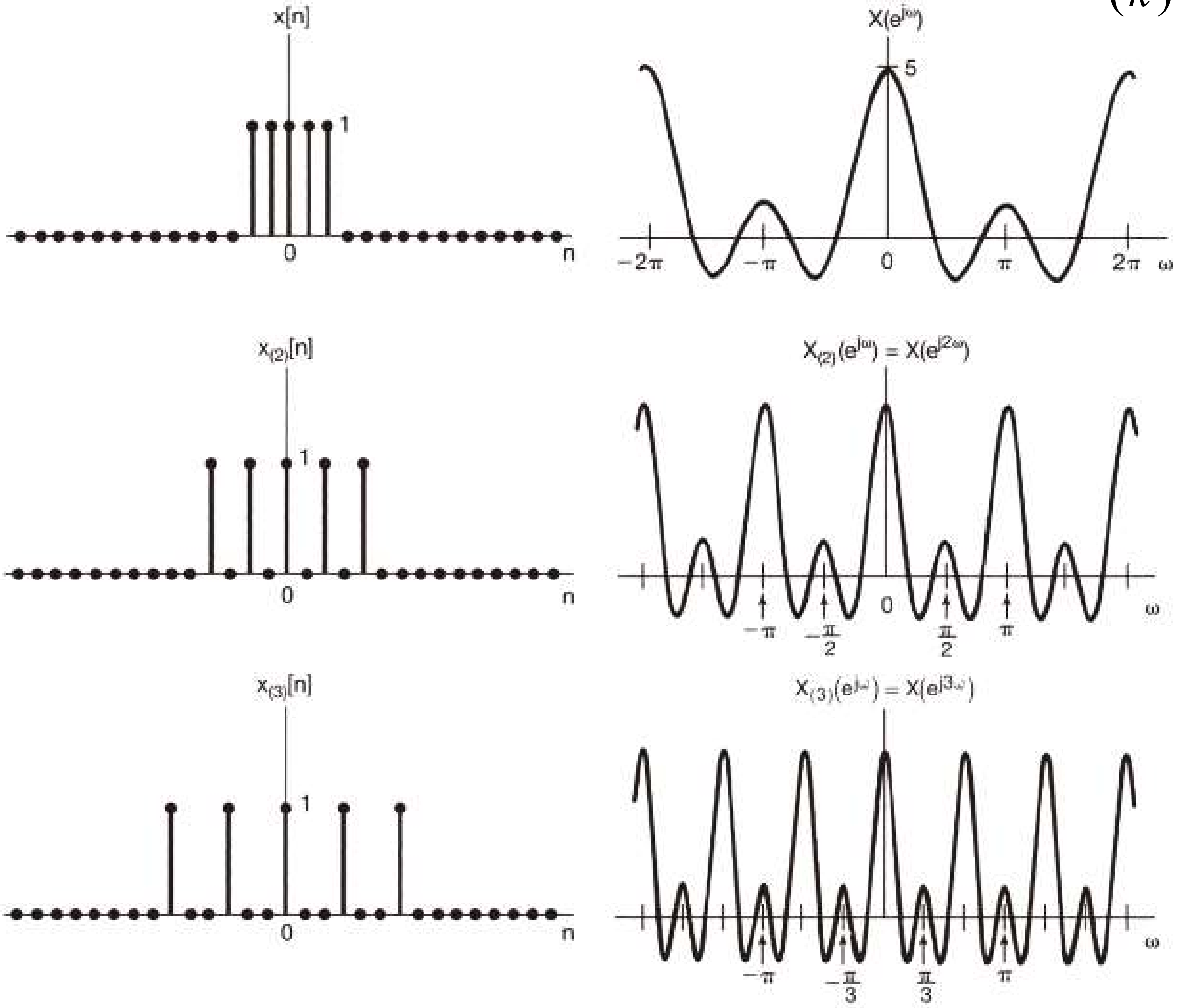
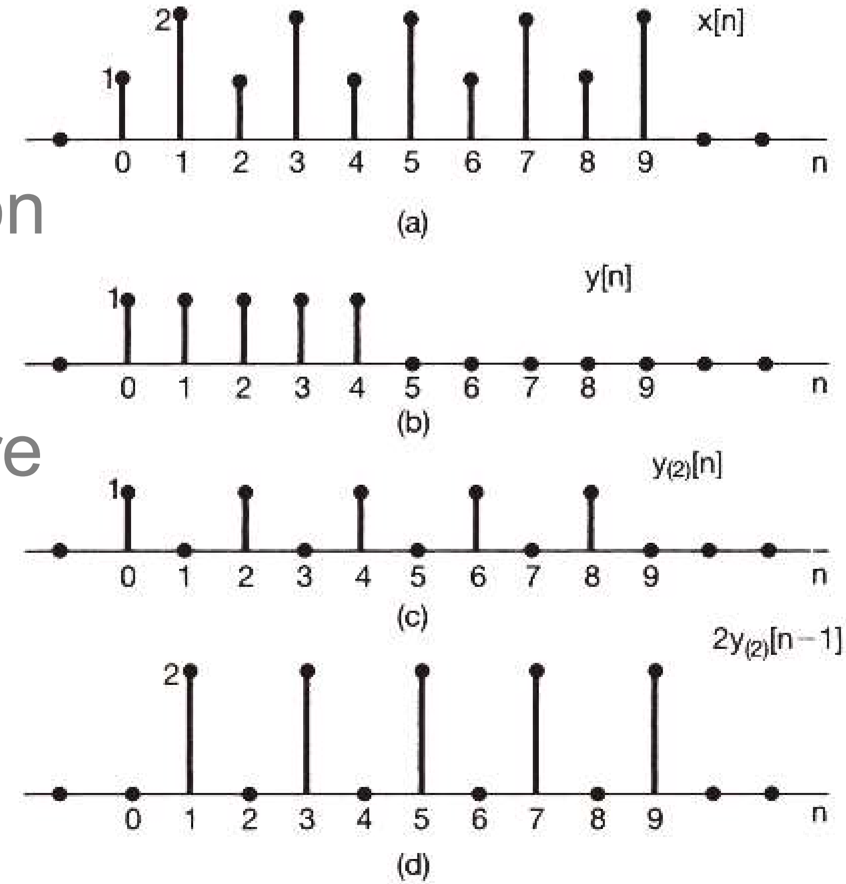


Figure 5.14 Inverse relationship between the time and frequency domains: As k increases, $x_{(k)}[n]$ spreads out while its transform is compressed.

Example 5.9

As an illustration of the usefulness of the time-expansion property in determining Fourier transforms, let us consider the sequence *x[n]* displayed in figure 5.15(a). This sequence can be related to the simpler sequence *y[n]* depicted in Figure 5.15(b). In particular



$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

where

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$g[n]=\begin{cases}1, & |n|<N_1 \\ 0, & |n|>N_1\end{cases}. \qquad G(e^{j\omega})=\frac{\sin\omega(N_1+1/2)}{\sin(\omega/2)}$$

$$x[n]\overset{F}{\longleftrightarrow}X(e^{j\omega}),$$

$$x[n-n_0]\overset{F}{\longleftrightarrow}e^{-j\omega n_0}X(e^{j\omega})$$

Next, note that $y[n]=g[n-2]$, where *g[n*] is a rectangular pulse as considered in Example 5.3 (with N_1 =2) and as depicted in figure 5.6(a). Consequently, from Example 5.3 and the time-shifting property, we see that

$$Y(e^{j\omega})=e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

$$y_{(k)}[n] \stackrel{F}{\longleftrightarrow} Y(e^{jk\omega}). \qquad Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}.$$

Using the time-expansion property, we then obtain

$$y_{(2)}[n] \stackrel{F}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)},$$

and using the linearity and time-shifting properties, we get

$$2y_{(2)}[n-1] \stackrel{F}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}.$$

$$y[n-n_0] \stackrel{F}{\longleftrightarrow} e^{-j\omega n_0}Y(e^{j\omega})$$

Combining these two results, we have

$$X(e^{j\omega}) = e^{-j4\omega}(1+2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right).$$

5.3.8 Differentiation in Frequency

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.(5.9)$$

Again, let

$$x[n] \overset{F}{\longleftrightarrow} X(e^{j\omega}).$$

If we use the definition of $X(e^{j\omega})$ in the analysis equation (5.9) and differentiate both sides, we obtain

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} -jnx[n]e^{-j\omega n}.$$

The right-hand side of this equation is the Fourier transform of $-jnx[n]$. Therefore, multiplying both sides by j, we see that

$$nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}. \qquad (5.46)$$

頻域微分性質

5.3.9 Parseval's Relation

If $x[n]$ and $X(e^{j\omega})$ are a Fourier transform pair, then

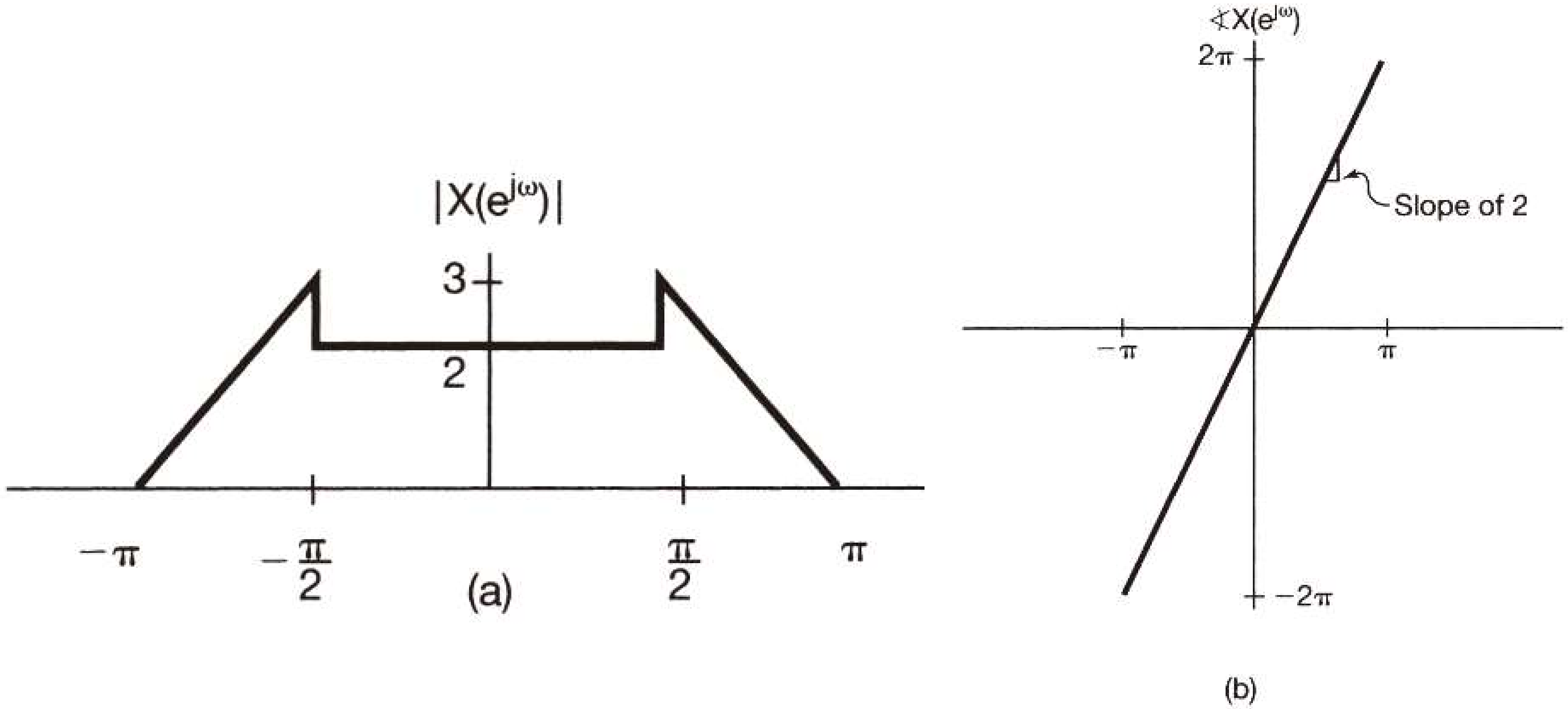
$$\sum_{n=-\infty}^{+\infty} \left| x[n] \right|^2 = \frac{1}{2\pi} \int_{2\pi} \left| X(e^{j\omega}) \right|^2 d\omega. \tag{5.47}$$

Parseval's relation states that this energy can also be determined by integrating the energy per unit frequency, $\left| X(e^{j\omega}) \right|^2 / 2\pi$, over a full 2π interval of distinct discrete-time frequencies.

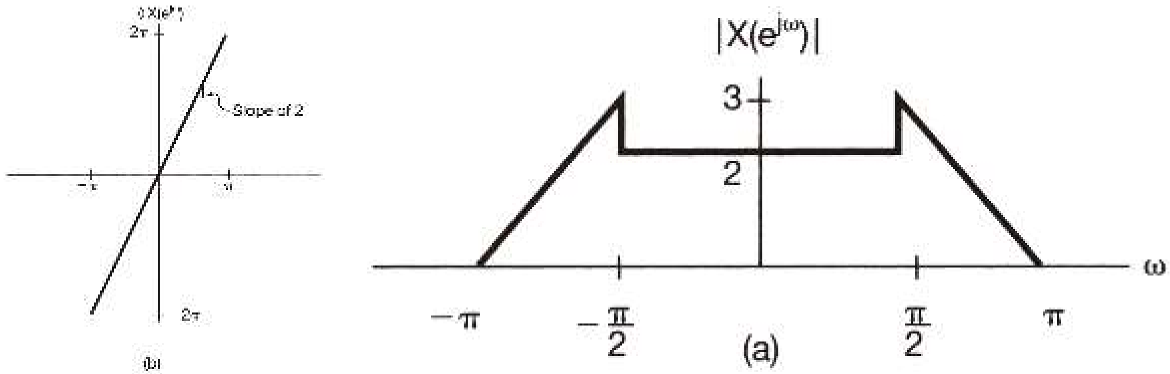
巴斯瓦關係式說明了訊號的能量亦可在頻域中積分求得。

Example 5.10

Consider the sequence *x[n]* whose Fourier transform $X(e^{j\omega})$ is depicted for $-\pi \leq \omega \leq \pi$ in figure 5.16. We wish to determine whether or not, in the time domain, *x[n]* is periodic, real, even, and/or of finite energy.



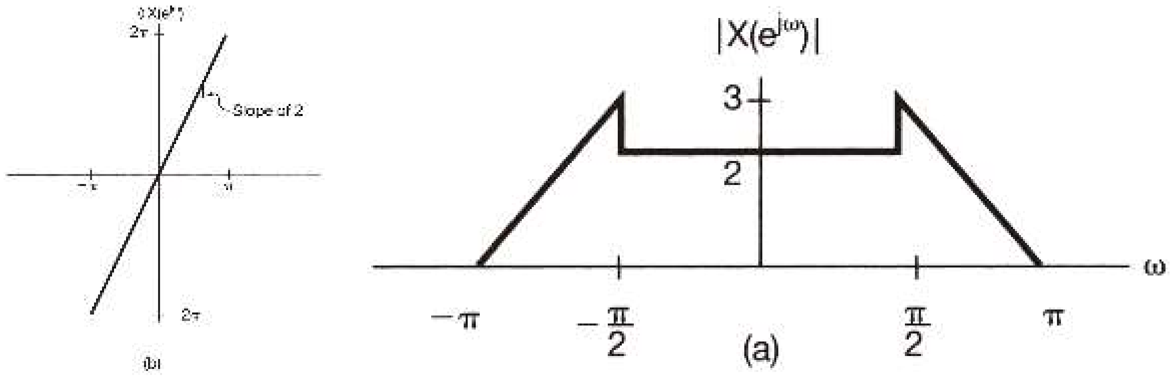
Example 5.10



Accordingly, we note first that periodicity in the time domain implies that the Fourier transform is zero, except possibly for impulses located at various integer multiples of the fundamental frequency. This is not true for $X(e^{j\omega})$. We conclude, then, that *x[n]* is not periodic.

Next, from the symmetry properties for Fourier transforms, we know that a real-valued sequence must have a Fourier transform of even magnitude and a phase function that is odd. This is true for the given $|X(e^{j\omega})|$ and $\measuredangle X(e^{j\omega})$. We thus conclude that *x[n]* is real.

Example 5.10



Third, if *x[n]* is an even function, then, by the symmetry properties for real signals, $X(e^{j\omega})$ must be real and even. However, since $X(e^{j\omega}) = \left|X(e^{j\omega})\right| e^{-j2\omega}$, $X(e^{j\omega})$ is not a real-valued function. Consequently, *x[n]* is not even.

Finally, to test for the finite-energy property, we may use Parseval's relation,

$$\sum_{n=-\infty}^{\infty} \left|x[n]\right|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega.$$

It is clear from figure 5.16 that integrating $\left|X(e^{j\omega})\right|^2$ from $-\pi$ to π will yield a finite quantity. We conclude that *x[n]* has finite energy.

5.4 The convolution Property

Specifically, if *x[n],h[n],* and *y[n]* are the input, impulse response, and output, respectively, of an LTI system, so that

對於一LTI系統：

$$y[n] = x[n] * h[n],$$

輸出、輸入及脈衝響應的迴旋積分關係(時域)

then

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}), \qquad (5.48)$$

輸出、輸入的傅立葉轉換與頻率響應關係(頻域)

Example 5.11

Consider an LTI system with impulse response

$$h[n] = \delta[n - n_0].$$

The frequency response is

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

Thus, for any input *x[n]* with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega}). \qquad (5.49)$$

Example 5.11

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega}).(5.49)$$

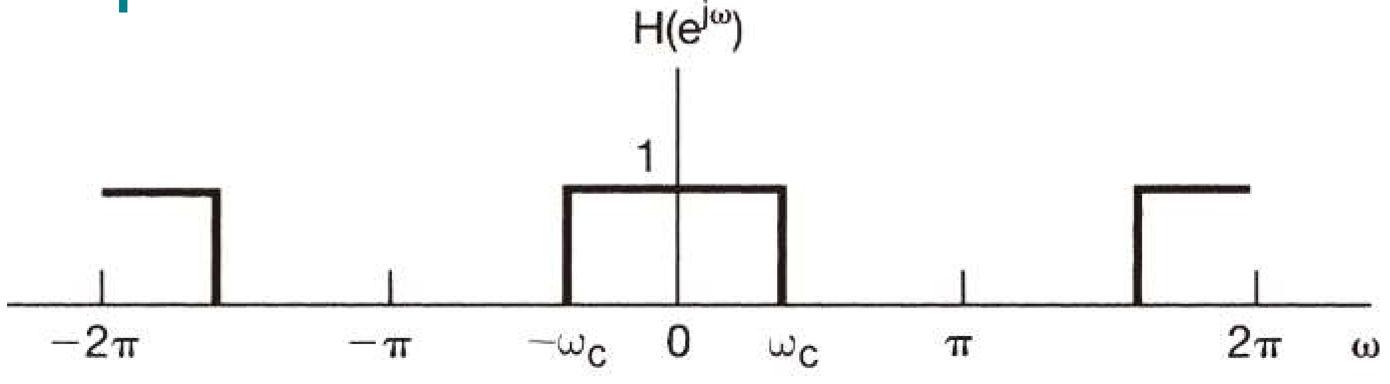
$\Updownarrow$

We note that, for this example, $y[n] = x[n - n_0]$ and eq. (5.49) is consistent with the time-shifting property. Note also that the frequency response

$$H(e^{j\omega}) = e^{-j\omega n_0}$$

of a pure time shift has unity magnitude at all frequencies and a phase characteristic $-\omega n_0$ that is **linear** with frequency.

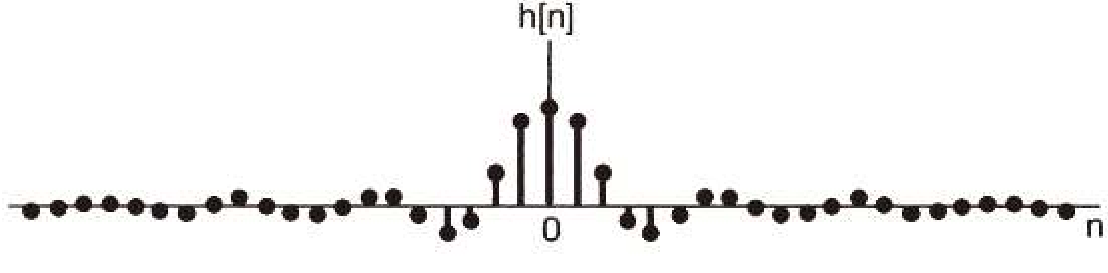
Example 5.12



(a)

Consider the discrete-time ideal lowpass filter in Section 3.9.2. This system has the frequency response $H(e^{j\omega})$ (Fig. 5.17(a)). Since the impulse response and frequency response of an LTI system are a Fourier transform pair, we can determine the impulse response of the ideal lowpass filter from the frequency response using the Fourier transform synthesis equation (5.8).

Example 5.12



(b)

In particular, using $-\pi \leq \omega \leq \pi$ as the interval of integration in that equation, we see from Figure 5.17(a) that

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega}) d^{j\omega n} d\omega = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega \qquad (5.50)$$

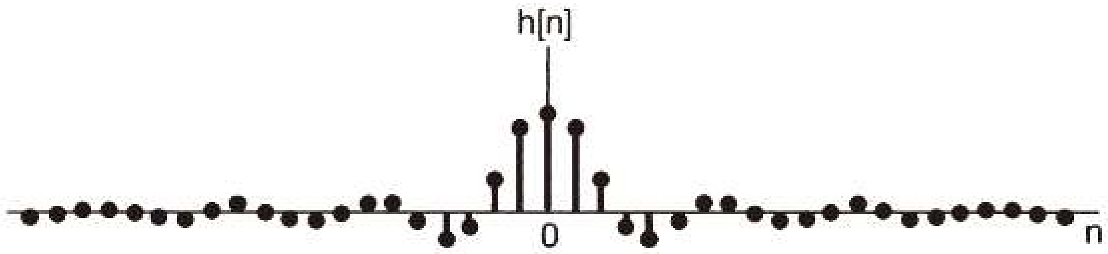
$$= \frac{\sin \omega_c n}{\pi n},$$

in figure 5.17(b).

5.4.1 Examples

In figure 5.17, we come across many of the same **issues** that surfaced with the continuous-time ideal lowpass filter in Example 4.18.

For a causal LTI system: $h[n] = 0 \quad \text{for } n < 0$



(b)

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} = \sum_{k=0}^{n-1} ar^k = a\,\frac{1-r^n}{1-r},$$

Example 5.13

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}.(5.9)$$

Consider an LTI system with impulse response

$$h[n] = \alpha^n u[n],$$

With $|\alpha| < 1$, and suppose that the input to this system is

$$x[n] = \beta^n u[n],$$

Example 5.13

With $|\beta| < 1$. Evaluating the Fourier transforms of *h[n]* and *x[n]*, we have

$$H(e^{j\omega}) = \frac{1}{1-\alpha e^{-j\omega}} \quad (5.51)$$

and

$$X(e^{j\omega}) = \frac{1}{1-\beta e^{-j\omega}}, \quad (5.52)$$

so that

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}. \quad (5.53)$$

Example 5.13 $Y(e^{j\omega}) = \dfrac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$.

As with Example 4.19, determining the inverse transform of $Y(e^{j\omega})$ is most easily done by expanding $Y(e^{j\omega})$ by the method of **partial fractions**. Specifically, $Y(e^{j\omega})$ is a ratio of polynomials in powers of $e^{-j\omega}$, and we would like to express this as a **sum of simpler terms** of this type so that we can find the inverse transform of each term by inspection (together, perhaps, with the use of the frequency differentiation property of Section 5.3.8).

Example 5.13

The general algebraic procedure for **rational transforms** is described in the appendix. For this example, if α≠β, the partial fraction expansion of $Y(e^{j\omega})$ is of the form

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}. \qquad (5.54)$$

Equating the right-hand sides of eqs (5.53) and (5.54), we find that

$$A = \frac{\alpha}{\alpha-\beta}, \qquad \beta = -\frac{\beta}{\alpha-\beta}.$$

$$\frac{A}{1-\alpha e^{-j\omega}}+\frac{B}{1-\beta e^{-j\omega}}=\frac{1}{(1-\alpha e^{-j\omega})(1-\beta e^{-j\omega})}$$

- Get A: multiply both size by $(1-\alpha e^{-j\omega})$

$$A+\frac{B(1-\alpha e^{-j\omega})}{1-\beta e^{-j\omega}}=\frac{1}{(1-\beta e^{-j\omega})}, \quad set \quad e^{-j\omega}=\frac{1}{\alpha}$$

$$A+\frac{B(1-1)}{1-\beta/\alpha}=\frac{1}{(1-\beta/\alpha)} \Rightarrow A=\frac{\alpha}{(\alpha-\beta)}$$

- Get B: multiply both size by $(1-\beta e^{-j\omega})$

$$\frac{A(1-\beta e^{-j\omega})}{1-\alpha e^{-j\omega}}+B=\frac{1}{(1-\alpha e^{-j\omega})}, \quad set \quad e^{-j\omega}=\frac{1}{\beta}$$

$$B=\frac{\beta}{(\beta-\alpha)}=-\frac{\beta}{(\alpha-\beta)}$$

Example 5.13

$$Y(e^{j\omega}) = \frac{A}{1-\alpha e^{-j\omega}} + \frac{B}{1-\beta e^{-j\omega}}.$$

$$A = \frac{\alpha}{\alpha-\beta}, \qquad \beta = -\frac{\beta}{\alpha-\beta}.$$

Therefore, from Example 5.1 and the linearity property, we can obtain the inverse transform of eq. (5.54) by inspection:

$$y[n] = \frac{\alpha}{\alpha-\beta}\alpha^n u[n] - \frac{\beta}{\alpha-\beta}\beta^n u[n] \qquad (5.55)$$

$$= \frac{1}{\alpha-\beta}\left[\alpha^{n+1}u[n] - \beta^{n+1}u[n]\right].$$

Example 5.13

For $\alpha = \beta$, the partial-fraction expansion in eq. (5.54) is not valid. However, in this case,

$$Y(e^{j\omega}) = \left(\frac{1}{1-\alpha e^{-j\omega}}\right)^2,$$

which can be expressed as

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right). \qquad (5.56)$$

Example 5.13 $nx[n] \overset{F}{\longleftrightarrow} j\frac{dX(e^{j\omega})}{d\omega}$.(5.46)

As in Example 4.19, we can use the frequency differentiation property, eq. (5.46), together with the Fourier transform pair

$$\alpha^n u[n] \overset{F}{\longleftrightarrow} \frac{1}{1-\alpha e^{-j\omega}},$$

to conclude that

$$n\alpha^n u[n] \overset{F}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right).$$

Example 5.13

$$n\alpha^n u[n] \stackrel{F}{\longleftrightarrow} j\frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right).$$

$$x[n-n_0] \stackrel{F}{\longleftrightarrow} e^{-j\omega n_0} X(e^{j\omega})$$

$$Y(e^{j\omega}) = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right).$$

To account for the factor $e^{j\omega}$, we use the time-shifting property to obtain

$$(n+1)\alpha^{n+1} u[n+1] \stackrel{F}{\longleftrightarrow} je^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1-\alpha e^{-j\omega}}\right),$$

and finally, accounting for the factor $1/\alpha$, in eq. (5.56), we obtain

$$y[n] = (n+1)\alpha^n u[n+1]. \qquad (5.57)$$

Example 5.13

It is worth noting that, although the right-hand side is multiplied by a step that begins at *n* = -1, the sequence $(n+1)\alpha^n u[n+1]$ is still zero prior to *n* = 0, since the factor *n* + 1 is zero at *n* = -1. Thus, we can alternatively express *y[n]* as

$$y[n] = (n+1)\alpha^n u[n]. \qquad (5.58)$$

5.4.1 Not every LTI system has a $H(e^{j\omega})$

The LTI system with impulse response $h[n] = 2^n u[n]$ dose not have a finite response to sinusoidal inputs, which is reflected in the fact that the Fourier transform analysis equation for *h[n]* diverges. However, if an LTI system is stable, then, from Section 2.3.7, its impulse response is absolutely summable; that is,

$$\sum_{n=-\infty}^{+\infty} |h[n]| < \infty$$

並非每一個離散時間LTI系統都具有頻率響應。

若一LTI系統為穩定，則其脈衝響應為絕對可加的，故對於穩定系統，頻率響應必定是收斂的。

5.5 The Multiplication Property

Consider *y[n]* equal to the product of $x_1[n]$ and $x_2[n]$, with $Y(e^{j\omega})$, $X_1(e^{j\omega})$, and $X_2(e^{j\omega})$ denoting the corresponding Fourier transforms. Then

若 $y[n] = x_1[n]x_2[n]$ (在時域中為相乘)

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n},$$

or since

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta, \qquad (5.60)$$

it follows that

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}. \qquad (5.61)$$

5.5 The Multiplication Property

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}.$$

Interchanging the order of summation and integration, we obtain

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})\left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta. \qquad (5.62)$$

The bracketed summation is $X_2(e^{j(\omega-\theta)})$, and consequently, eq. (5.62) becomes (periodic convolution)

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta. \qquad (5.63)$$

則 $Y(e^{j\omega})$ 如 (5.63)式(在頻域中為迴旋運算)。

5.5 The Multiplication Property

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta.$$

Equation (5.63) corresponds to a periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, and the integral in this equation can be evaluated over any interval of length 2π.

(5.63)式為一週期性迴旋積分，積分區間寬度為 2π。

Example 5.15

Consider the problem of finding the Fourier transform $X(e^{j\omega})$ of a signal *x[n]* which is the product of two other signals; that is,

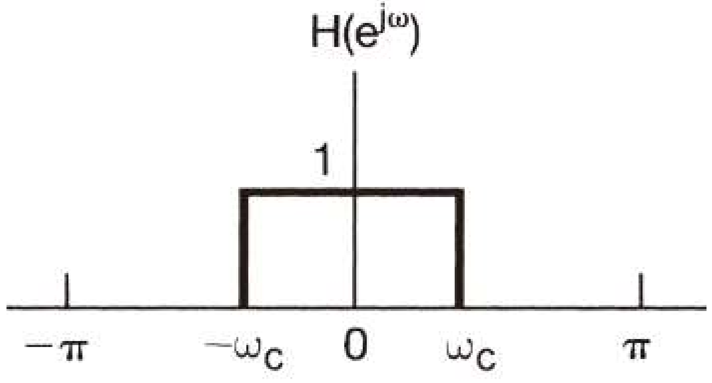
$$x[n] = x_1[n]x_2[n],$$

where

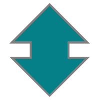
$$x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$$

and

$$x_2[n] = \frac{\sin(\pi n/2)}{\pi n}.$$



(a)



$$h[n] = \frac{\sin \omega_c n}{\pi n}$$

Example 5.15

From the multiplication property given in eq. (5.63), we know that $X(e^{j\omega})$ is the periodic convolution of $X_1(e^{j\omega})$ and $X_2(e^{j\omega})$, where the integral in eq. (5.63) can be taken over any interval of length 2π. Choosing the interval $-\pi \langle \theta \le \pi$, we obtain

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\omega})X_2(e^{j(\omega-\theta)})d\theta. \qquad (5.64)$$

Example 5.15

Equation (5.64) resembles aperiodic convolution, except for the fact that the integration is limited to the interval $-\pi < \theta \leq \pi$. However, we can convert the equation into an ordinary convolution by defining

$$\hat{X}_1(e^{j\omega}) = \begin{cases} X_1(e^{j\omega}) & for -\pi < \omega \leq \pi \\ 0 & otherwise \end{cases}.$$

Example 5.15

Then, replacing $X_1(e^{j\theta})$ in eq. (5.64) by $\hat{X}_1(e^{j\theta})$, and using the fact that $\hat{X}_1(e^{j\theta})$ is zero for $|\theta| > \pi$, we see that

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta.$$

Example 5.15

$$X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta}) X_2(e^{j(\omega-\theta)}) d\theta.$$

Thus, $X(e^{j\omega})$ is 1/(2π) times the aperiodic convolution of the rectangular pulse $\hat{X}_1(e^{j\omega})$ and the periodic square wave $X_2(e^{j\omega})$, both of which are shown in Figure 5.19. $X(e^{j\omega})$ is shown in Figure 5.20.

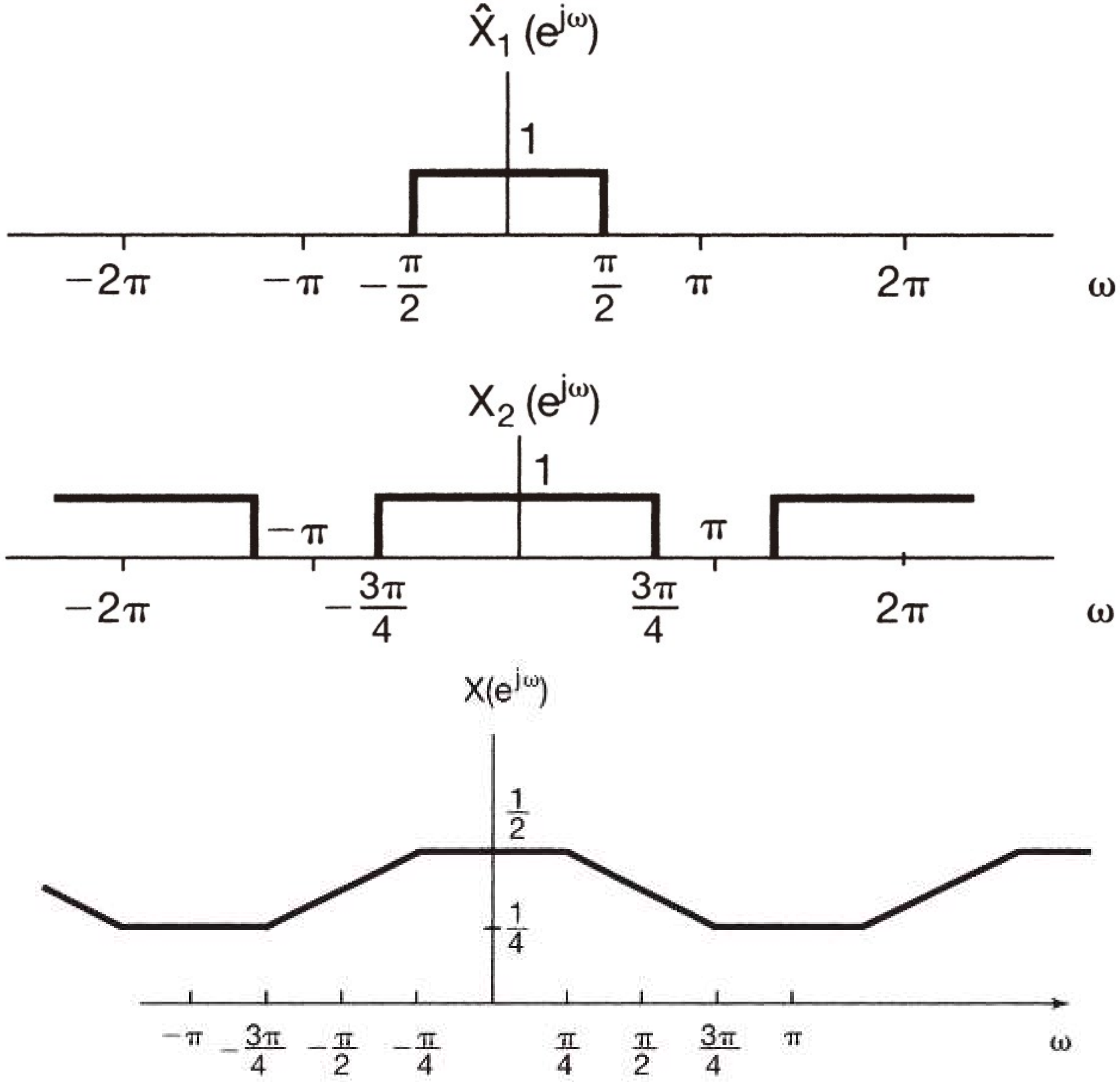


Figure 5.20 Result of the periodic convolution in Example 5.15.

TABLE 5.1 PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM 表 5.1 離散時間傅立葉轉換的性質

Section	Property	Aperiodic Signal	Fourier Transform
		$x[n]$ $y[n]$	$\left.\begin{matrix} X(e^{j\omega}) \\ Y(e^{j\omega}) \end{matrix}\right\}$ periodic with period 2π
5.3.2	Linearity	$ax[n] + by[n]$	$aX(e^{j\omega}) + bY(e^{j\omega})$
5.3.3	Time Shifting	$x[n - n_0]$	$e^{-j\omega n_0}X(e^{j\omega})$
5.3.3	Frequency Shifting	$e^{j\omega_0 n}x[n]$	$X(e^{j(\omega-\omega_0)})$
5.3.4	Conjugation	$x^*[n]$	$X^*(e^{-j\omega})$
5.3.6	Time Reversal	$x[-n]$	$X(e^{-j\omega})$
5.3.7	Time Expansion	$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n = \text{multiple of } k \\ 0, & \text{if } n \neq \text{multiple of } k \end{cases}$	$X(e^{jk\omega})$
5.4	Convolution	$x[n] * y[n]$	$X(e^{j\omega})Y(e^{j\omega})$
5.5	Multiplication	$x[n]y[n]$	$\frac{1}{2\pi}\int_{2\pi} X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$
5.3.5	Differencing in Time	$x[n] - x[n-1]$	$(1 - e^{-j\omega})X(e^{j\omega})$
5.3.5	Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - e^{-j\omega}}X(e^{j\omega})$ $+\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$
5.3.8	Differentiation in Frequency	$nx[n]$	$j\frac{dX(e^{j\omega})}{d\omega}$
5.3.4	Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} X(e^{j\omega}) = X^*(e^{-j\omega}) \\ \mathcal{R}e\{X(e^{j\omega})\} = \mathcal{R}e\{X(e^{-j\omega})\} \\ \mathcal{I}m\{X(e^{j\omega})\} = -\mathcal{I}m\{X(e^{-j\omega})\} \\ \lvert X(e^{j\omega})\rvert = \lvert X(e^{-j\omega})\rvert \\ \sphericalangle X(e^{j\omega}) = -\sphericalangle X(e^{-j\omega}) \end{cases}$
5.3.4	Symmetry for Real, Even Signals	$x[n]$ real an even	$X(e^{j\omega})$ real and even
5.3.4	Symmetry for Real, Odd Signals	$x[n]$ real and odd	$X(e^{j\omega})$ purely imaginary and odd
5.3.4	Even-odd Decomposition of Real Signals	$x_e[n] = \mathcal{E}v\{x[n]\}$ $[x[n]$ real$]$ $x_o[n] = \mathcal{O}d\{x[n]\}$ $[x[n]$ real$]$	$\mathcal{R}e\{X(e^{j\omega})\}$ $j\mathcal{I}m\{X(e^{j\omega})\}$
5.3.9	Parseval's Relation for Aperiodic Signals		

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(e^{j\omega})|^2 d\omega$$

TABLE 5.2 BASIC DISCRETE-TIME FOURIER TRANSFORM PAIRS　表 5.2　基本的離散時間傅立葉轉換對

Signal	Fourier Transform	Fourier Series Coefficients (if periodic)
$\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	a_k
$e^{j\omega_0 n}$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\cos \omega_0 n$	$\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi m}{N}$ $a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$\sin \omega_0 n$	$\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$	(a) $\omega_0 = \frac{2\pi r}{N}$ $a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ (b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic
$x[n] = 1$	$2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$	$a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$
Periodic square wave $x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$ and $x[n+N] = x[n]$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$ $a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$
$\sum_{k=-\infty}^{+\infty} \delta[n - kN]$	$\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$	$a_k = \frac{1}{N}$ for all k
$a^n u[n], \; \lvert a\rvert < 1$	$\frac{1}{1 - ae^{-j\omega}}$	—
$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$	$\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$	—
$\frac{\sin Wn}{\pi n} = \frac{W}{\pi} \operatorname{sinc}\left(\frac{Wn}{\pi}\right)$ $0 < W < \pi$	$X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$ $X(\omega)$ periodic with period 2π	—
$\delta[n]$	1	—
$u[n]$	$\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$	—
$\delta[n - n_0]$	$e^{-j\omega n_0}$	—
$(n+1)a^n u[n], \; \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^2}$	—
$\frac{(n+r-1)!}{n!(r-1)!} a^n u[n], \; \lvert a\rvert < 1$	$\frac{1}{(1 - ae^{-j\omega})^r}$	—

5.7.1 Duality in the Discrete-Time Fourier Series

Recall duality in CTFT

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega(4.8)$$

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt(4.9)$$

duality is not as clear in DTFT

$$x[n]=\frac{1}{2\pi}\int_{2\pi}X(e^{j\omega})e^{j\omega n}d\omega,$$

$$X(e^{j\omega})=\sum_{n=-\infty}^{+\infty}x[n]e^{-j\omega n}.(5.9)$$

Clear duality in DTFS

$$x[n]=\sum_{k=\langle N\rangle}a_k e^{jk(2\pi/N)n}.$$

$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle}x[n]e^{-jk(2\pi/N)n}.$$

$$x[n]=\sum_{k=\langle N\rangle}a[k]e^{jk(2\pi/N)n}.$$

$$a[k]=\frac{1}{N}\sum_{n=\langle N\rangle}x[n]e^{-jk(2\pi/N)n}.$$

5.7.1 Duality in the Discrete-Time Fourier Series

$$x[n] = \sum_{k=\langle N\rangle} a[k] e^{jk(2\pi/N)n} . (synthesis)$$

$$x[n] \overset{FS}{\longleftrightarrow} a[k]$$

$$a[k] = \frac{1}{N} \sum_{n=\langle N\rangle} x[n] e^{-jk(2\pi/N)n} . (analysis)$$

$$n = -m; x[-m] = \sum_{k=\langle N\rangle} a[k] e^{-jk(2\pi/N)m}$$

$$a[k] \overset{FS}{\longleftrightarrow} \frac{1}{N} x[-m]$$

$$\frac{1}{N} x[-m] = \frac{1}{N} \sum_{k=\langle N\rangle} a[k] e^{-jm(2\pi/N)k}$$

$$a[n] \overset{FS}{\longleftrightarrow} \frac{1}{N} x[-k]$$

$$(analysis)$$

5.7.1 Duality in the Discrete-Time Fourier Series

See that the pair of properties

$$x[n-n_0] \stackrel{FS}{\longleftrightarrow} a_k e^{-jk(2\pi/N)n_0} \qquad (5.68)$$

and

$$e^{jm(2\pi/N)n} x[n] \stackrel{FS}{\longleftrightarrow} a_{k-m} \qquad (5.69)$$

are dual. Similarly, from the same table, we can extract another pair of dual properties;

$$\sum_{r=\langle N\rangle} x[r]y[n-r] \stackrel{FS}{\longleftrightarrow} N a_k b_k \qquad (5.70)$$

and

$$x[n]y[n] \stackrel{FS}{\longleftrightarrow} \sum_{l=\langle N\rangle} a_l b_{k-l}. \qquad (5.71)$$

Example 5.16

$$square \overset{FS}{\longleftrightarrow} \sin c$$

$$\sin c \overset{FS}{\longleftrightarrow} square$$

Consider the following periodic signal with a period of N = 9:

$$x[n] = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq multiple \quad of \quad 9 \\ \dfrac{5}{9}, & n = multiple \quad of \quad 9 \end{cases}$$

(5.72)

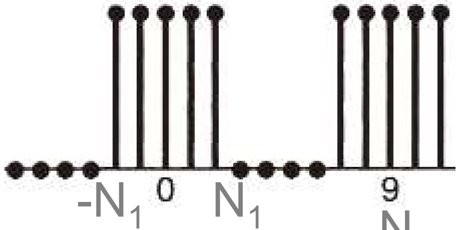


Figure 3.16

$$a_k = \begin{cases} \dfrac{1}{N}\dfrac{\sin(2\pi k(N_1+1/2)/N)}{\sin(\pi k/N)}, & for \ \ k \neq 0, \pm N, \pm 2N, \ldots \\ \dfrac{2N_1+1}{N}, & for \ \ k = 0, \pm N, \pm 2N, \ldots \end{cases} . \quad x[n] = \begin{cases} \dfrac{1}{9}\dfrac{\sin(5\pi n/9)}{\sin(\pi n/9)}, \\ \dfrac{5}{9}, \end{cases}$$

In Chapter 3, we found that a rectangular square wave has Fourier coefficients in a form much as in eq. (5.72). Duality, then, suggests that the coefficients for *x[n]* must be in the form of a rectangular square wave. To see this more precisely, let *g[n]* be a rectangular square wave with period *N* = 9 such that

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4. \end{cases}$$

Example 5.16

$$g[n] = \begin{cases} 1, & |n| \le 2 \\ 0, & 2 < |n| \le 4. \end{cases}$$

The Fourier series coefficients b_k for *g[n]* can be determined from Example 3.12 as

$$b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \ne multiple \quad of \quad 9 \\ \frac{5}{9}, & k = multiple \quad of \quad 9 \end{cases}$$

$$g[n] \overset{FS}{\longleftrightarrow} b_k$$

The Fourier series analysis equation (3.95) for *g[n]* can now be written as

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-jk(2\pi/9)n}.$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}.$$

$$(analysis)$$

Example 5.16

$$b_k = \frac{1}{9}\sum_{n=-2}^{2}(1)e^{-jk(2\pi/9)n}.$$

Interchanging the names of the variables k and n and noting that $x[n] = b_n$, we find that

$$b_n = \frac{1}{9}\sum_{k=-2}^{2}(1)e^{-jn(2\pi/9)k} = x[n]$$

Letting $k' = -k$ in the sum on the right side, we obtain

$$x[n] = \frac{1}{9}\sum_{k'=-2}^{2} e^{+jn(2\pi/9)k'} = \sum_{k'=-2}^{2}\frac{1}{9}e^{+jk'(2\pi/9)n} (synthesis)$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}.(synthesis)$$

Example 5.16

$$x[n] = \sum_{k=-2}^{2} \frac{1}{9} e^{+jk(2\pi/9)n} (synthesis).$$

We thus conclude that the Fourier coefficients of *x[n]* are given by

$$a_k = \begin{cases} 1/9, & |k| \le 2 \\ 0, & 2 < |k| \le 4, \end{cases}$$

and, of course, are periodic with period $N = 9$.

$$x[n] \xleftrightarrow{FS} a[k] \qquad a[n] \xleftrightarrow{FS} \frac{1}{N} x[-k]$$

5.7.2 Duality between the discrete-Time Fourier Transform and the Continuous-Time Fourier Series

We repeat these equations here for convenience:

[eq. (5.8)] $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega, \qquad (5.73)$$

[eq. (5.9)] $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}, \qquad (5.74)$$

$$\Rightarrow \quad x[n] \overset{FT}{\longleftrightarrow} X(e^{j\omega})$$

[eq. (3.38)] $$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \qquad (5.75)$$

[eq. (3.39)] $$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt. \qquad (5.76)$$

$$\Rightarrow \quad x(t) \overset{FS}{\longleftrightarrow} a_k$$

5.7.2 Duality between the discrete-Time Fourier Transform and the Continuous-Time Fourier Series

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega,$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n},$$

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t},$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt.$$

Observation

$X(e^{j\omega})$ period 2π

$x(t)$ period T

$x[n]$ aperiodic

a_k aperiodic

$$x[-m] = \frac{1}{T}\int_T X(e^{j\omega})e^{-j\omega m}d\omega, T = 2\pi$$

$$X(e^{j\omega}) = \sum_{m=-\infty}^{+\infty} x[-m]e^{j\omega m}$$

$$\Rightarrow X(e^{j\omega}) \overset{FS}{\longleftrightarrow} x[-m]$$

5.7.2 Duality between the discrete-Time Fourier Transform and the Continuous-Time Fourier Series

TABLE 5.3 SUMMARY OF FOURIER SERIES AND TRANSFORM EXPRESSIONS

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ **continuous frequency** **aperiodic in frequency**	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$ **discrete time** **aperiodic in time**	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

duality (continuous time: time domain ↔ frequency domain, Fourier Transform); duality (CT Fourier Series frequency domain ↔ DT Fourier Transform time domain); duality (discrete time: time domain ↔ frequency domain, Fourier Series)

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

A general linear constant-coefficient difference equation for an LTI system with input *x[n]* and output *y[n]* is of the form

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]. \qquad (5.78)$$

*N*階線性常係數差分方程一般式

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

There are two ways to find the frequency response of a LTI system：

I. eigenfunction: $x[n]=e^{j\omega n} \longrightarrow Y[n]=H(e^{j\omega})e^{j\omega n}$

II. The convolution property, eq. (5.48), of the discrete-time Fourier transform then implies that

$$Y(e^{j\omega})=H(e^{j\omega})X(e^{j\omega})$$

$$\Downarrow$$

$$H(e^{j\omega})=\frac{Y(e^{j\omega})}{X(e^{j\omega})}. \qquad (5.79)$$

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

$$F\{\sum_{k=0}^{N} a_k y[n-k]\} = F\{\sum_{k=0}^{M} b_k x[n-k]\}$$

Applying the Fourier transform to both sides of eq. (5.78) and using the **linearity** and **time-shifting** properties, we obtain the expression

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y(e^{j\omega}) = \sum_{k=0}^{M} b_k e^{-jk\omega} X(e^{j\omega}),$$

or equivalently,

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}. \qquad (5.80)$$

Example 5.19

$$H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}.$$

Consider a causal LTI system that is characterized by the difference equation

$$y[n] - \frac{3}{4} y[n-1] + \frac{1}{8} y[n-2] = 2x[n]. \qquad (5.84)$$

From eq. (5.80), the frequency response is

$$H(e^{j\omega}) = \frac{2}{1 - \frac{3}{4} e^{-j\omega} + \frac{1}{8} e^{-j2\omega}}. \qquad (5.85)$$

Example 5.19

As a first step in obtaining the impulse response, we factor the denominator of eq. (5.85):

$$H(e^{j\omega}) = \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}. \qquad (5.86)$$

$H(e^{j\omega})$ can be expanded by the method of partial fractions, as in Example A.3 in the appendix. The result of this expansion is

$$H(e^{j\omega}) = \frac{4}{1-\frac{1}{2}e^{-j\omega}} - \frac{2}{1-\frac{1}{4}e^{-j\omega}}. \qquad (5.87)$$

Example 5.19

$a^n u[n], \quad |a| < 1$ | $\dfrac{1}{1 - ae^{-j\omega}}$

$$H(e^{j\omega}) = \frac{4}{1 - \frac{1}{2}e^{-j\omega}} - \frac{2}{1 - \frac{1}{4}e^{-j\omega}}.$$

The inverse transform of each term can be recognized by inspection, with the result that

$$h[n] = 4\left(\frac{1}{2}\right)^n u[n] - 2\left(\frac{1}{4}\right)^n u[n]. \qquad (5.88)$$

5.8 Systems Characterized by Linear Constant-Coefficient Difference Equations

Specifically, after expanding $H(e^{j\omega})$ by the method of **partial fractions**, we can find the inverse transform of each term by inspection. The same approach can be applied to the frequency response of any LTI system described by a linear constant-coefficient difference equation in order to determine the system impulse response.

若將 $H(e^{j\omega})$部份分式展開後，可得其反傅立葉轉換，亦即此LTI系統的脈衝響應。

Example 5.20

Consider the LTI system of Example 5.19, and let the input to this system be

$$x[n] = \left(\frac{1}{4}\right)^n u[n].$$

Then, using eq. (5.80) and Example 5.1 or 5.18, we obtain

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \left[\frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-j\omega})}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right] \qquad (5.89)$$

$$= \frac{2}{(1-\frac{1}{2}e^{-j\omega})(1-\frac{1}{4}e^{-e\omega})^2}.$$

Example 5.20

As described in the appendix, the form of the partial-fraction expansion in this case is

$$Y(e^{j\omega}) = \frac{B_{11}}{1-\frac{1}{4}e^{-j\omega}} + \frac{B_{12}}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{B_{21}}{1-\frac{1}{2}e^{-j\omega}}, \qquad (5.90)$$

Example 5.20

where the constants B_{11} , B_{12} , and B_{21} can be determined using the techniques described in the Appendix. This particular expansion is worked out in detail in ExampleA.4, and the values obtained are

$$B_{11} = -4, \qquad B_{12} = -2, \qquad B_{21} = 8,$$

so that

$$Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}. \qquad (5.91)$$

Example 5.20

$$Y(e^{j\omega}) = -\frac{4}{1-\frac{1}{4}e^{-j\omega}} - \frac{2}{(1-\frac{1}{4}e^{-j\omega})^2} + \frac{8}{1-\frac{1}{2}e^{-j\omega}}.$$

$(n+1)a^n u[n], \quad \lvert a\rvert < 1$	$\frac{1}{(1-ae^{-j\omega})^2}$

The first and third terms are of the same type as those encountered in Example 5.19, while the second term is of the same form as one seen in Example **5.13**. Either from these examples or from Table 5.2, we can invert each of the terms in eq. (5.91) to obtain the inverse transform

$$y[n] = \left\{-4\left(\frac{1}{4}\right)^n - 2(n+1)\left(\frac{1}{4}\right)^n + 8\left(\frac{1}{2}\right)^n\right\}u[n]. \tag{5.92}$$

5.9 Summary

- Derive FT for aperiodic signal from FS
- Convergence of FT
- FT for periodic signals
- Properties of FT: linearity, time-shifting, Conjugate symmetry, differencing& accumulation, time-expansion, duality, Parseval’s relation, etc.
- Convolution & multiplication properties
- Duality in the Discrete-Time Fourier Series
- Solving Linear Constant-Coefficient Difference Equations using FT properties