Chapter 4
The continuous-time Fourier Transform

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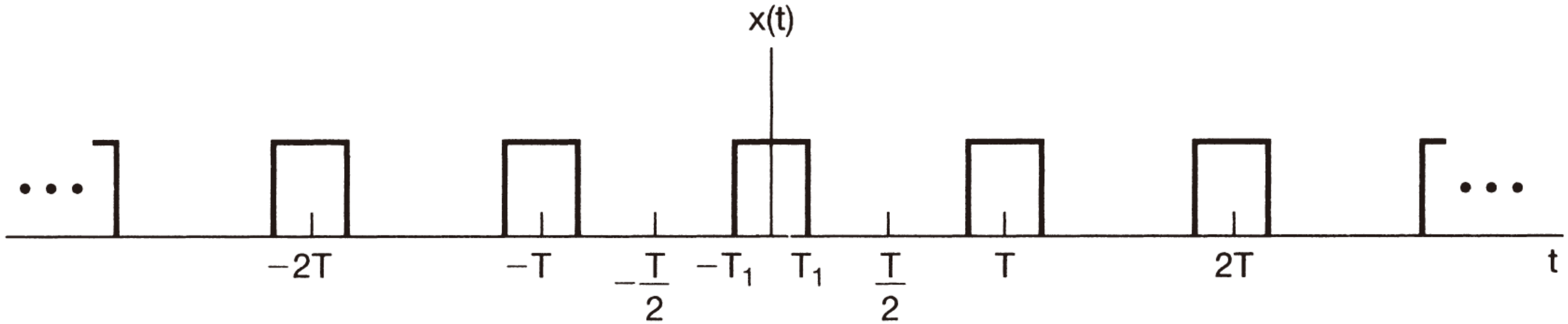
4.0 Introduction

- We extend fourier series for **periodic** signal to fourier transform for **periodic** and **aperiodic** signals with **finite energy**

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

To gain some insight into the nature of the Fourier transform representation, we begin by revisiting the Fourier series representation for the continuous-time periodic square wave examined in Example 3.5. Specifically, over one period,

$$x(t)=\begin{cases}1, & |t|<T_1\\ 0, & T_1<|t|<T/2\end{cases}$$



4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

As determined in Example 3.5, the Fourier series coefficients a_k for this square wave are

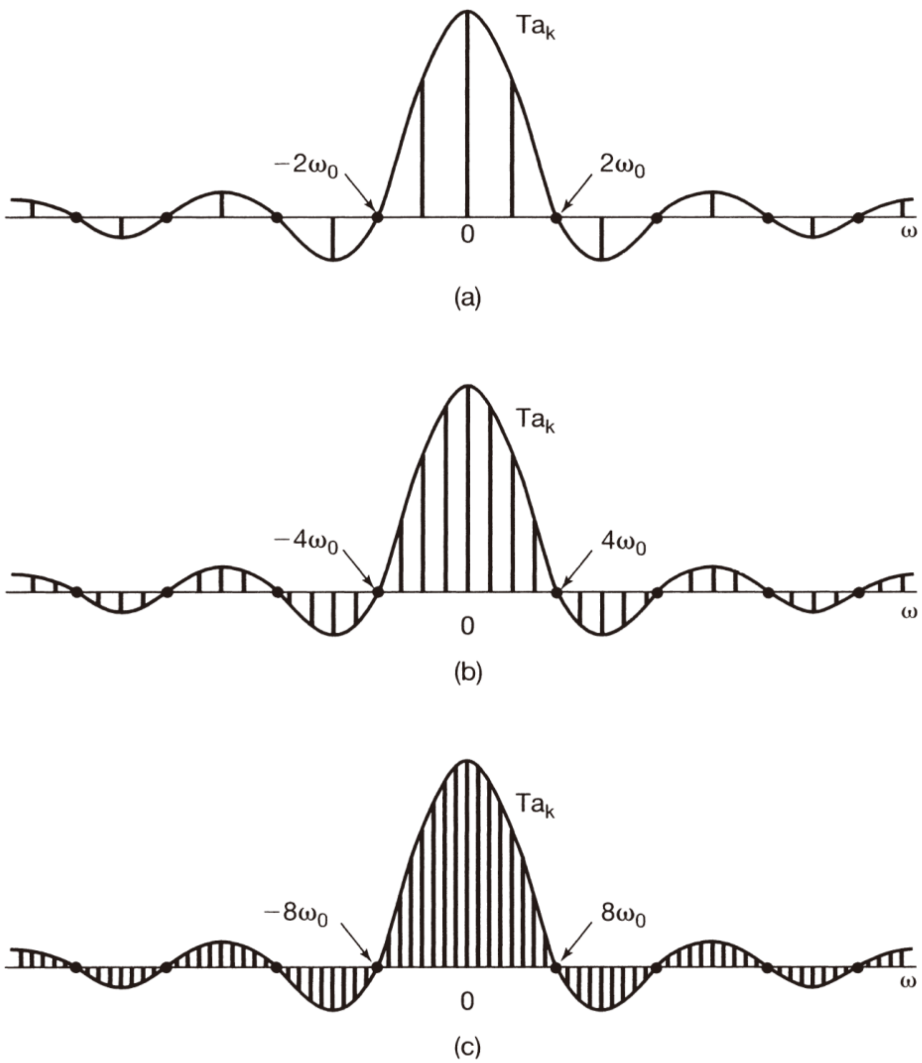
[eq.(3.44)] $$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} \qquad (4.1)$$

An alternative way of interpreting eq.(4.1) is as samples of an envelope function, specifically,

$$Ta_k = \frac{2\sin\omega T_1}{\omega}\bigg|_{\omega=k\omega_0}. \qquad (4.2)$$

and we refer $\dfrac{2\sin\omega T_1}{\omega}$ as it envelope

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal



$$\frac{2\sin\omega T_1}{\omega}$$

$$Ta_k = \frac{2\sin\omega T_1}{\omega}\bigg|_{\omega=k\omega_0}.$$

Now let's fix T_1 and change T from $4T_1$ (a), $8T_1$(b), $16T_1$(c)

- Envelope hasn't changes
- ω_0 becomes smaller
- Sample more densely

$$\omega_0 = \frac{2\pi}{T}$$

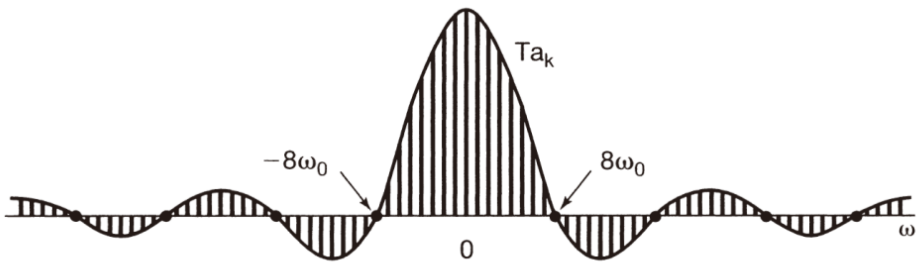
$$\omega T_1 = \pi$$

$$\omega = \frac{\pi}{T_1}$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

T increases, or equivalently, as the fundamental frequency $\omega_0 = 2\pi / T$ decreases, the envelope is sampled with a closer and closer spacing.

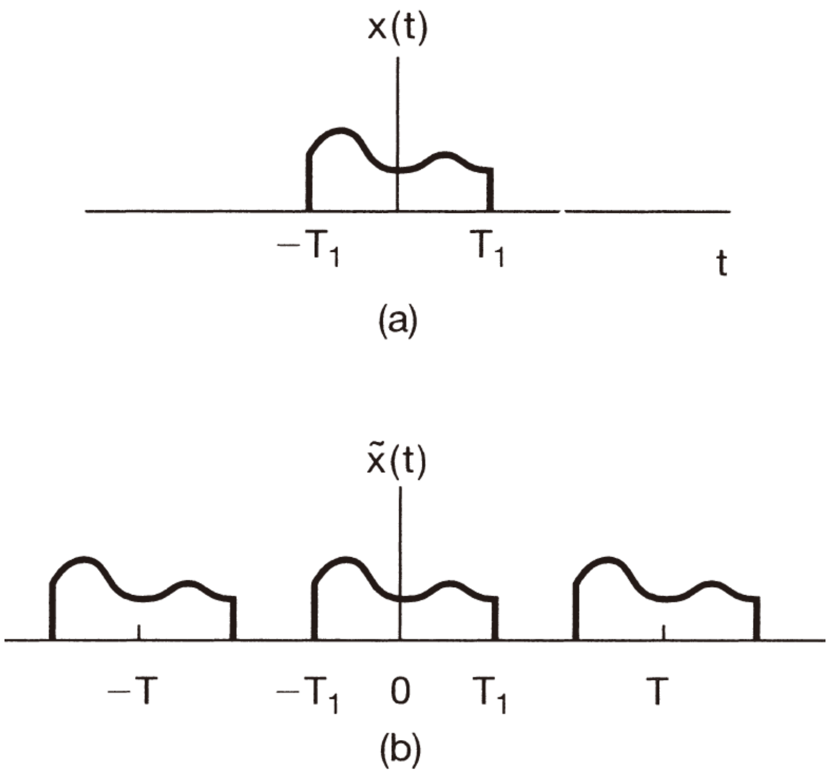
The set of Fourier series coefficients approaches the envelope function as $T \to \infty$.



(c)

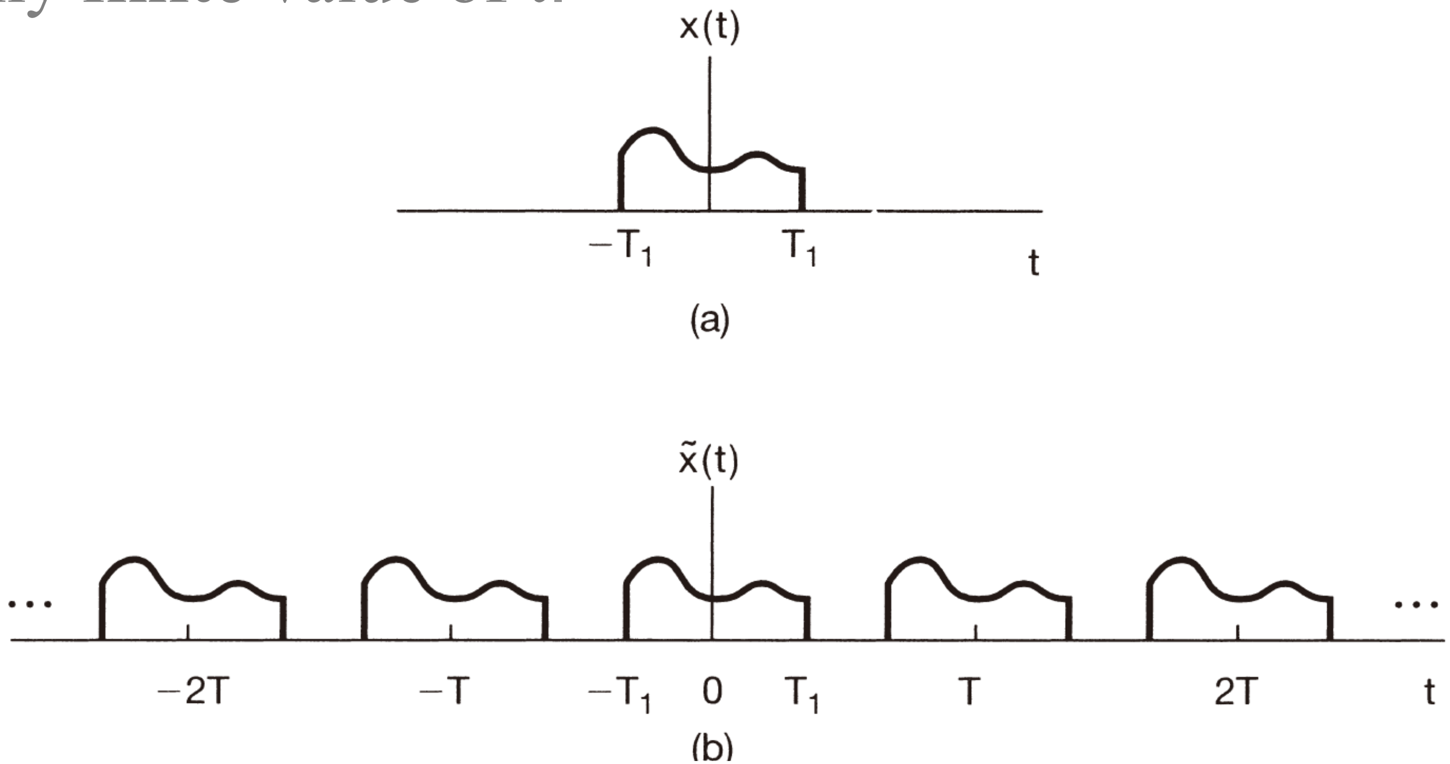
4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

In particular, consider a signal x(t) that is of **finite** duration. That is, for some number $T_1, x(t) = 0$ if $|t| > T_1$ as illustrated in Figure 4.3(a). From this aperiodic signal, we can construct a periodic signal $\tilde{x}(t)$ for which *x(t)* is one period, as indicated in Figure 4.3(b).



4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

As T becomes larger, $\tilde{x}(t)$ is identical to x(t) over a longer interval, and as $T \to \infty$, $\tilde{x}(t)$ it is equal to x(t) for any **finite** value of t.



4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

Recall from FS:

in eq. (3.39) carried out over the interval

$-T/2 \le t \le T/2$, we have

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}, \tag{4.3}$$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\omega_0 t} dt, \tag{4.4}$$

where $\omega_0 = 2\pi / T$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \tilde{x}(t)e^{-jk\omega_0 t}dt, (4.4)$$

Since $\tilde{x}(t) = x(t)$ for $|t| < T/2$, and also, since *x(t)*=0 outside this interval, eq. (4.4) can be rewritten as

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

4.1.1 Development of the Fourier Transform Representation of an Aperiodic Signal

$$a_k = \frac{1}{T}\int_{-\infty}^{+\infty} x(t)e^{-jk\omega_0 t}dt.$$

Therefore, defining the envelope $X(j\omega)$ of Ta_k as

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt, \tag{4.5}$$

we have, for the coefficients a_k,

$$a_k = \frac{1}{T}X(jk\omega_0). \tag{4.6}$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (4.3) \qquad a_k = \frac{1}{T} X(jk\omega_0) \quad (4.6)$$

Combining eqs. (4.6) and (4.3), we can express $\tilde{x}(t)$ in terms of $X(j\omega)$ as

$$\tilde{x}(t) = \sum_{k=-\infty}^{+\infty} \frac{1}{T} X(jk\omega_0) e^{jk\omega_0 t},$$

or equivalently, since $2\pi / T = \omega_0$,

$$\tilde{x}(t) = \frac{1}{2\pi} \sum_{k=-\infty}^{+\infty} X(jk\omega_0) e^{jk\omega_0 t} \omega_0. \tag{4.7}$$

4.1.1Development of the Fourier Transform Representation of an Aperiodic Signal

$$\tilde{x}(t) = \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 (4.7), \omega_0 = \frac{2\pi}{T}$$

As $T \rightarrow \infty$, $\tilde{x}(t)$ approaches *x(t),* and consequently, in the limit eq. (4.7) becomes a representation of *x(t).*

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$ Inverse Fourier Transform (4.8)

and

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.$$ Fourier Transform(4.9=4.5)

Spectrum of x(t)

4.1.2 Convergence of Fourier Transforms

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt. (4.9)$$

In 4.1.1, we derive Fourier Transform pairs for aperiodic signal with **finite** duration. Acutally, it is also valid for signal of **infinite** duration

Consider $X(j\omega)$ evaluated according to eq. (4.9), and let $\hat{x}(t)$ denote the signal obtained by using in the right-hand side of eq. (4.8). That is,

$$\hat{x}(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega.$$

4.1.2 Convergence of Fourier Transforms

If *x(t)* has finite energy, i.e., if it is square integrable, so that

$$\int_{-\infty}^{+\infty} |x(t)|^2 \, dt < \infty, \qquad (4.11)$$

then we are guaranteed that $X(j\omega)$ is finite [i.e., eq. (4.9) converges] and that ,with $e(t)$ denoting the error between $\hat{x}(t)$ and *x(t)* [i.e., $e(t) = \hat{x}(t) - x(t)$]

$$\int_{-\infty}^{+\infty} |e(t)|^2 \, dt = 0. \qquad (4.12)$$

*no energy in their difference.

4.1.2 Convergence of Fourier Transforms

- Alternative conditions:

These conditions, again referred to as the Dirichlet conditions, require that:

1. *x(t)* be absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |x(t)|dt < \infty. \tag{4.13}$$

2. *x(t)* have a finite number of maxima and minima within any finite interval.

3. *x(t)* have a finite number of discontinuities within any finite interval. Furthermore, each of these discontinuities must be finite.

4.1.3 Examples of Continuous-Time Fourier Transforms

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \quad (4.8)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt \quad (4.9)$$

Example 4.1

Consider the signal

$$x(t) = e^{-at}u(t) \qquad a > 0.$$

From eq. (4.9),

$$X(j\omega) = \int_0^{\infty} e^{-at}e^{-j\omega t}dt = -\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^{\infty}.$$

That is,

$$X(j\omega) = \frac{1}{a+j\omega}, \qquad a > 0.$$

Example 4.1

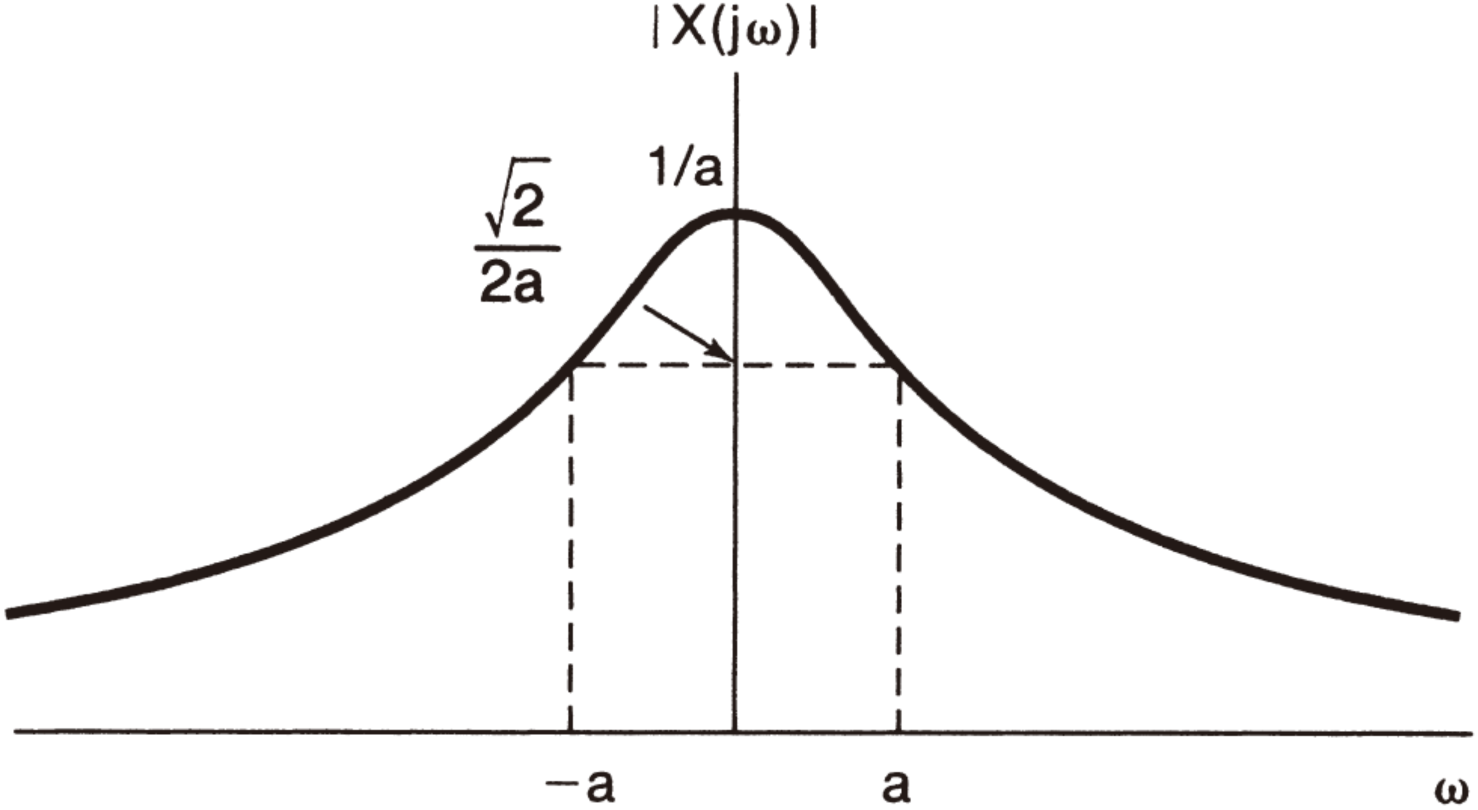
Since this Fourier transform is complex valued, to plot it as a function of ω , we express $X(j\omega)$ in terms of its magnitude and phase:

$$\left|X(j\omega)\right| = \frac{1}{\sqrt{a^2+\omega^2}}, \qquad \measuredangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

Each of these components is sketched in figure 4.5.

Example 4.1

$$|X(j\omega)| = \frac{1}{\sqrt{a^2 + \omega^2}}$$



(a)

Example 4.1

$$\measuredangle X(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right).$$

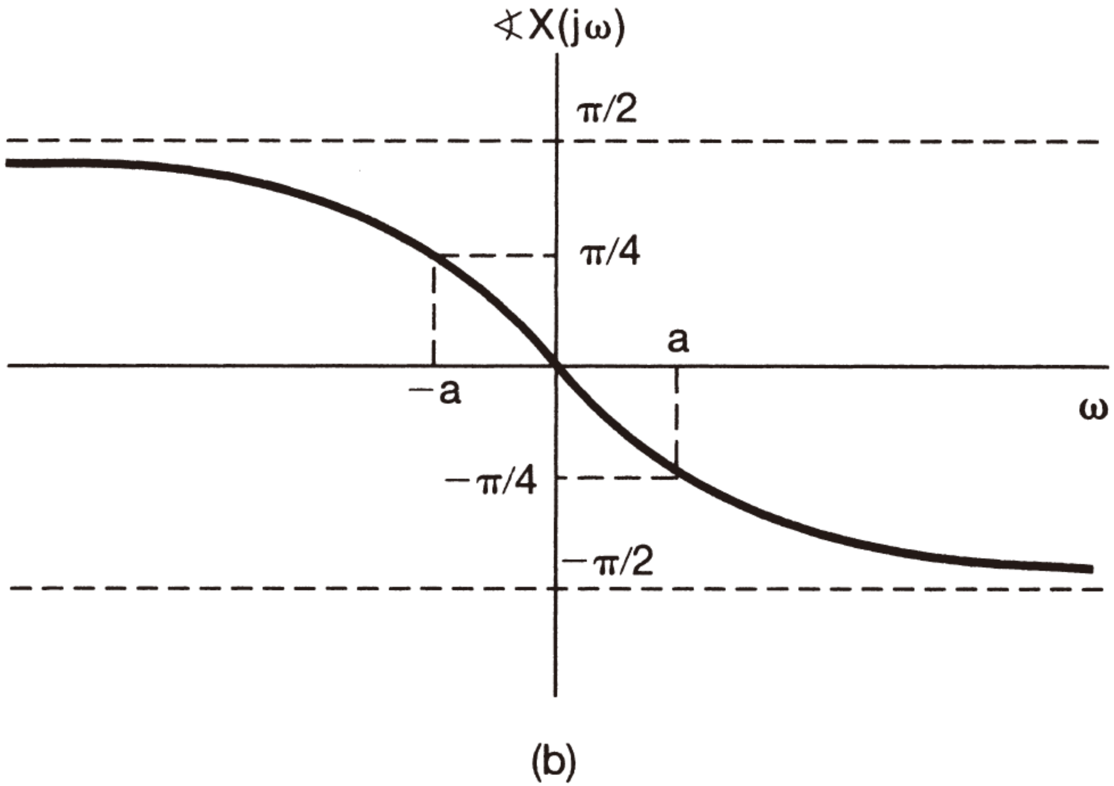


Figure 4.5 Fourier transform of the signal $x(t) = e^{-at}u(t), a > 0$, considered in Example 4.1.

Example 4.1

$$X(j\omega)=\int_0^\infty e^{-at}e^{-j\omega t}dt=-\frac{1}{a+j\omega}e^{-(a+j\omega)t}\Big|_0^\infty .$$

Note that if a is complex rather than real, then *x(t)* is absolutely integrable as long as $\Re e\{a\}>0$, and in this case the preceding calculation yields the same form for $X(j\omega)$. That is,

$$X(j\omega)=\frac{1}{a+j\omega},\qquad \Re e\{a\}>0$$

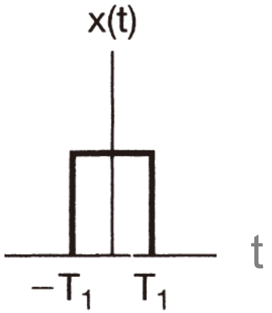
$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\omega)e^{j\omega t}d\omega$$

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt.$$

Example 4.4

Consider the rectangular pulse signal

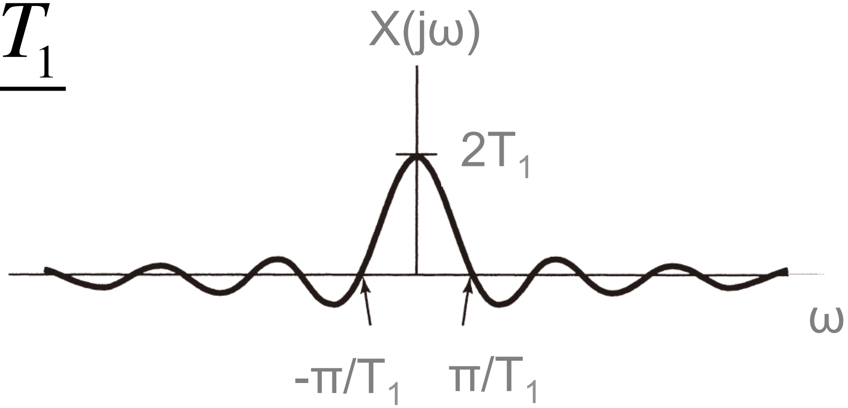
$$x(t)=\begin{cases}1, & |t|<T_1 \\ 0, & |t|>T_1\end{cases}. \qquad (4.16)$$



From Eq. 4.9, we obtain its Fourier transform as

$$X(j\omega)=\int_{-T_1}^{T_1}e^{-j\omega t}dt=2\frac{\sin\omega T_1}{\omega}$$

$$=2T_1\frac{\sin\omega T_1}{\omega T_1}.$$



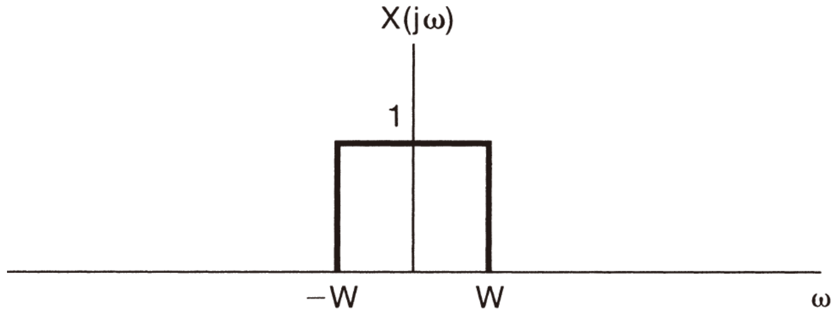
Example 4.5

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega$$

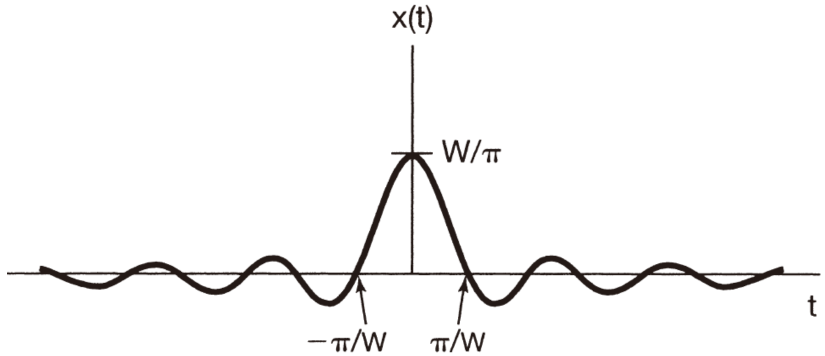
$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt.$$

Consider the signal *x(t)* whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} \quad .(4.18)$$



(a)



(b)

Using the synthesis equation (4.8), we can determine

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t}\,d\omega = \frac{\sin Wt}{\pi t} = \frac{W}{\pi}\frac{\sin Wt}{Wt} \quad .(4.19)$$

4.1.3 duality property

Fig. 4.8

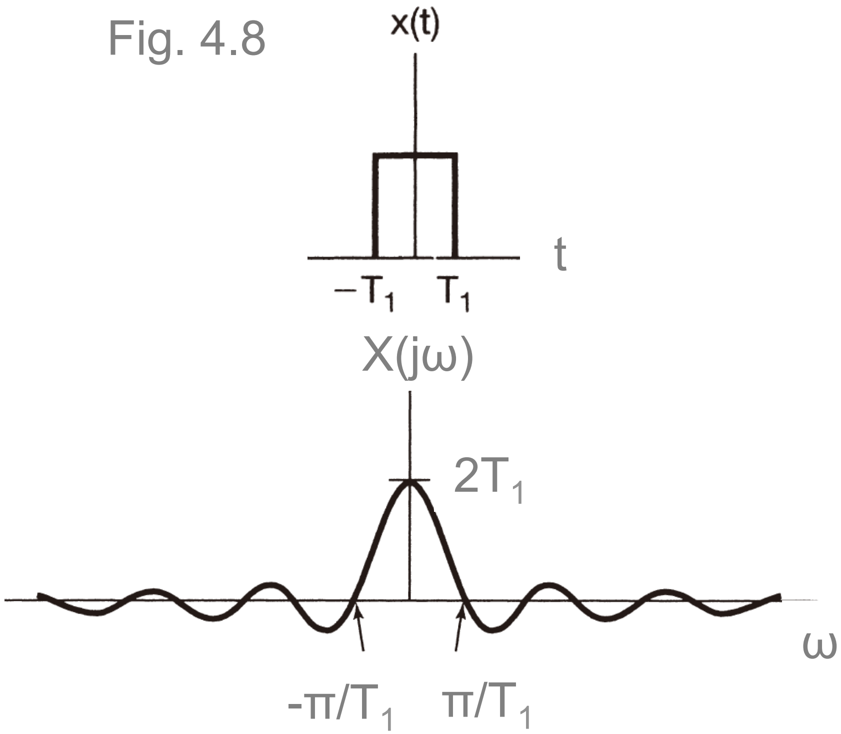
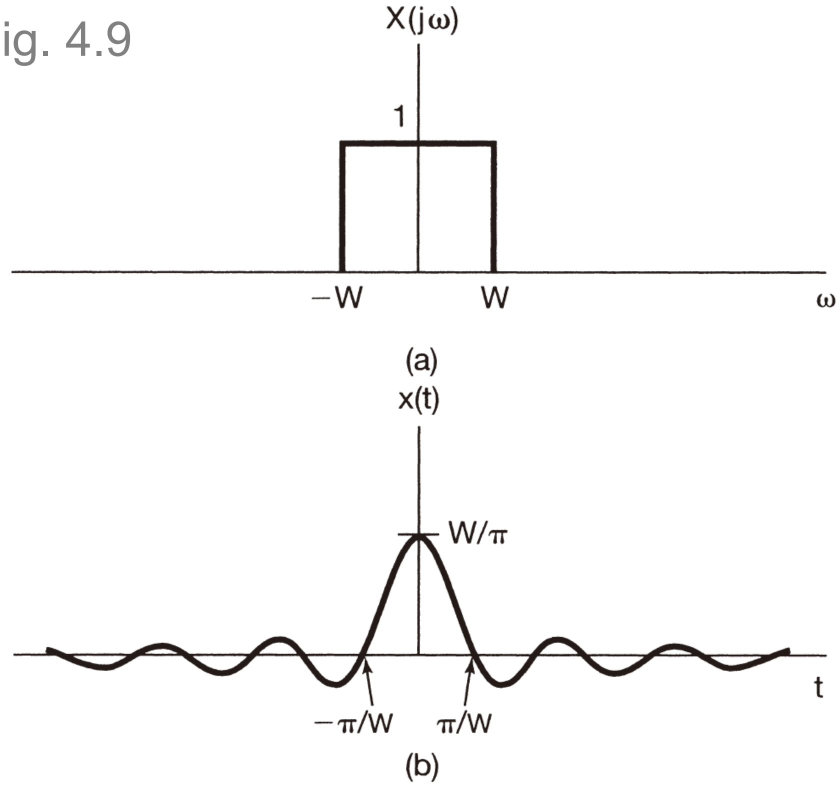


Fig. 4.9



Comparing Figures 4.8 and 4.9 or, equivalently, eqs. (4.16) and (4.17) with eqs. (4.18) and (4.19), we see an interesting relationship. In each case, the Fourier transform pair consists of a function of the form

$(\sin a\theta)/b\theta$ and a rectangular pulse.

4.1.3 Examples of Continuous-Time Fourier Transforms

A commonly used precise form for the sinc function is

$$\mathrm{sin}\,c(\theta) = \frac{\sin \pi\theta}{\pi\theta}. \qquad (4.20)$$

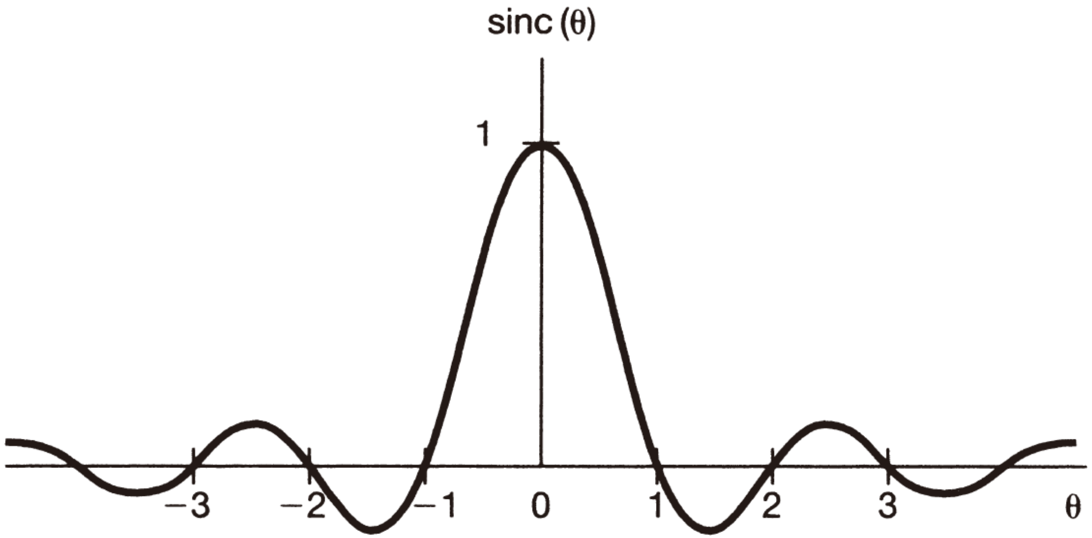
The sinc function is plotted in figure 4.10.

We can use sinc to express

eqs.(4.17) and (4.19)

$$\frac{2\sin \omega T_1}{\omega} = 2T_1 \,\mathrm{sin}\,c\left(\frac{\omega T_1}{\pi}\right)$$

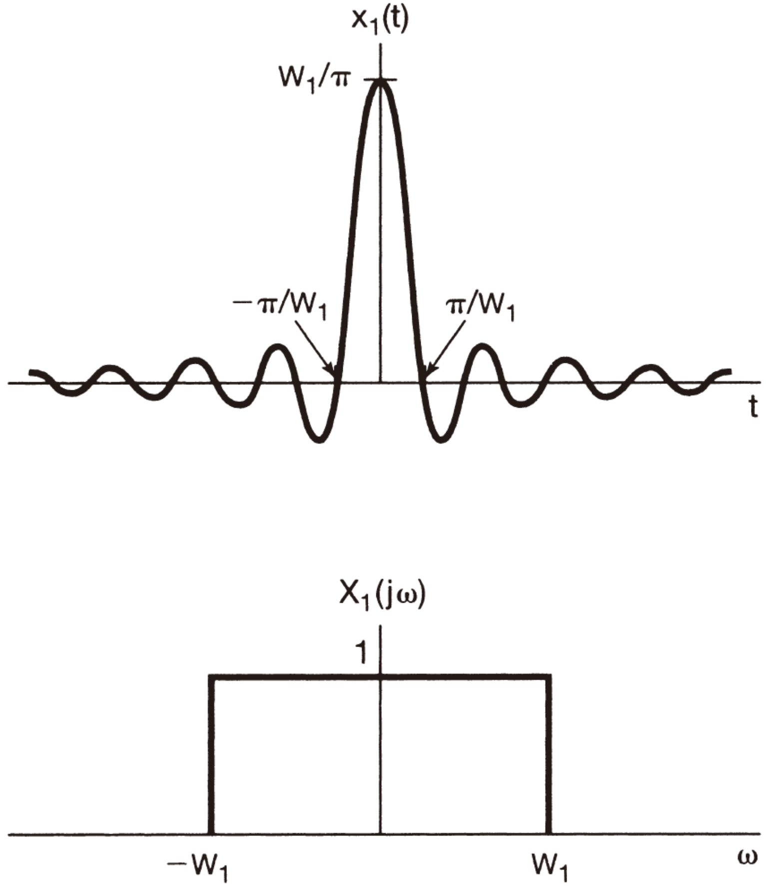
$$\frac{\sin Wt}{\pi t} = \frac{W}{\pi}\,\mathrm{sin}\,c\left(\frac{Wt}{\pi}\right).$$



4.1.3 Examples of Continuous-Time Fourier Transforms

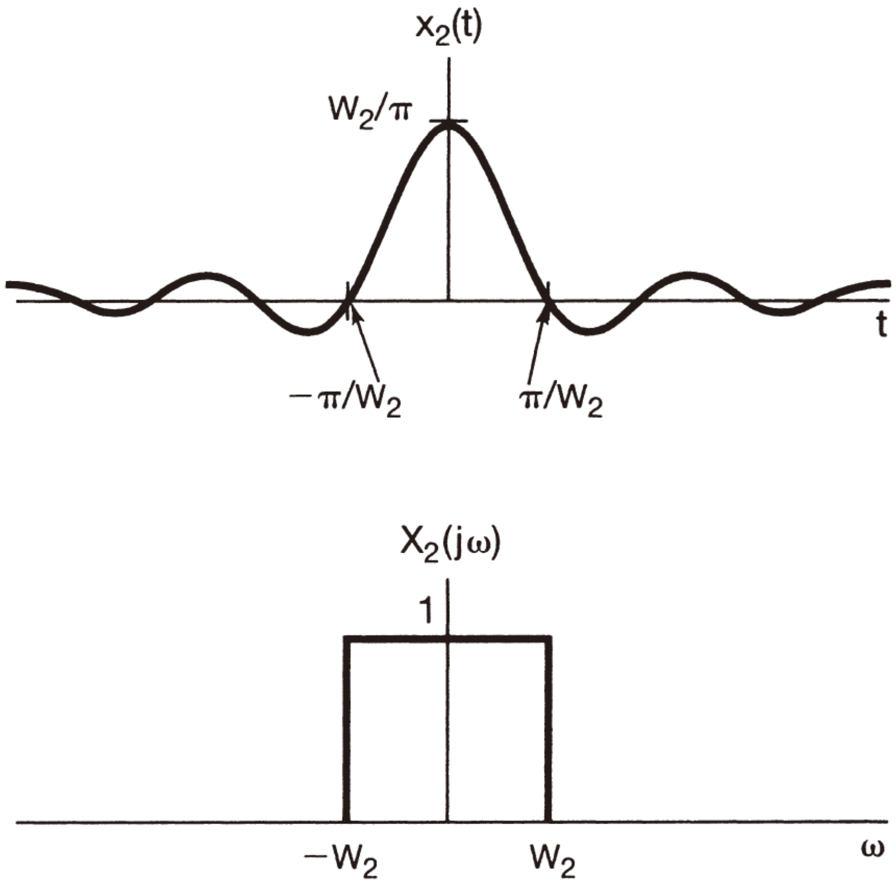
$$X(j\omega)=\begin{cases}1, & |\omega|<W_1 \\ 0, & |\omega|>W_1\end{cases} .$$

$$\frac{\sin W_1 t}{\pi t}=\frac{W_1}{\pi}\operatorname{sin}c\left(\frac{W_1 t}{\pi}\right).$$



(a)

4.1.3 Examples of Continuous-Time Fourier Transforms



(b)

4.1.3 Examples of Continuous-Time Fourier Transforms

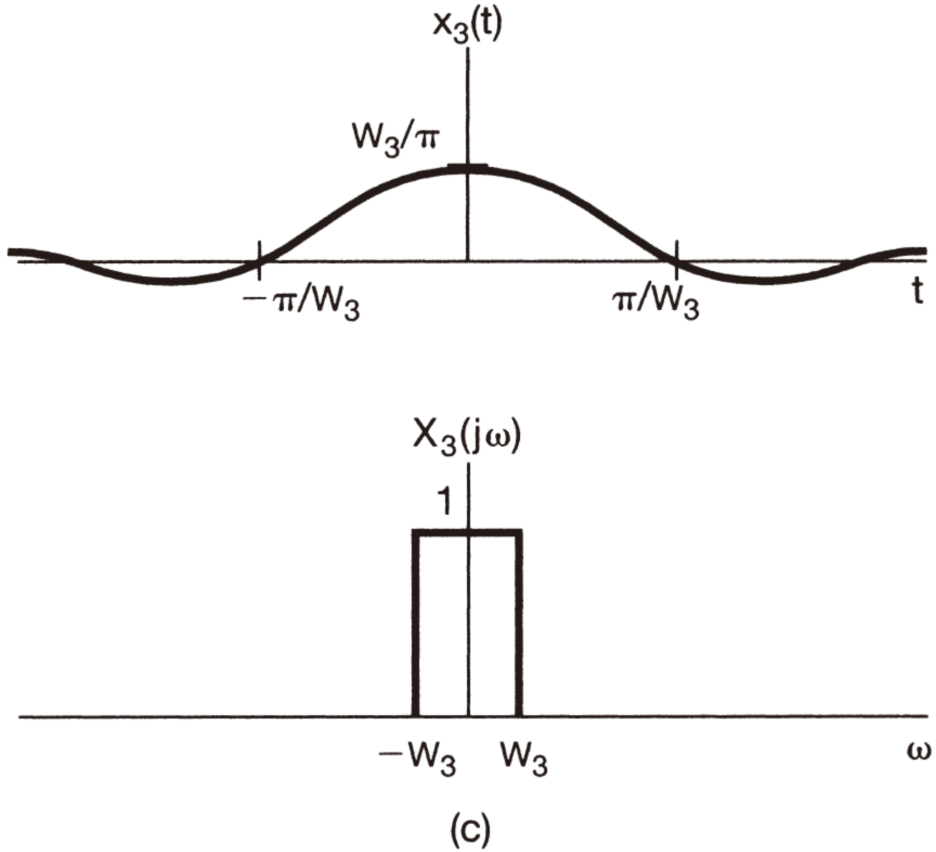


Figure 4.11 Fourier transform pair of Figure 4.9 for several different values of W.

4.2 The Fourier Transform for periodic signals

To suggest the general result, let us consider a signal *x(t)* with Fourier transform $X(j\omega)$ that is a single impulse of area 2π at $\omega = \omega_0$; that is,

$$X(j\omega) = 2\pi\delta(\omega - \omega_0). \tag{4.21}$$

To determine the signal *x(t)* for which this is the Fourier transform, we can apply the inverse transform relation, eq. (4.8), to obtain

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0)e^{j\omega t}d\omega = e^{j\omega_0 t}.$$

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega - \omega_0)$$

4.2 The Fourier Transform for periodic signals

$$e^{j\omega_0 t} \leftrightarrow 2\pi\delta(\omega-\omega_0)$$

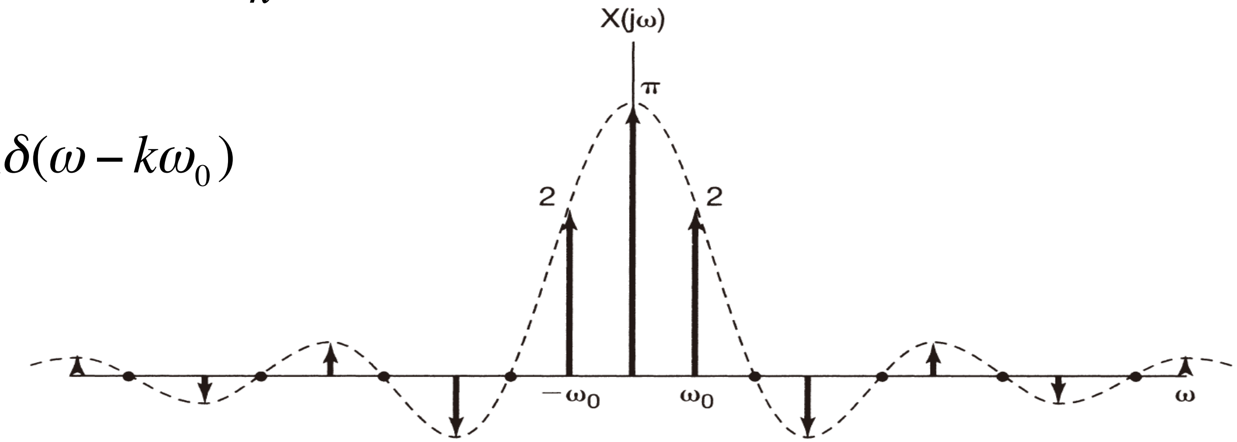
More generally, if $X(j\omega)$ is of the form of a linear combination of impulses equally spaced in frequency, that is,

$$X(j\omega)=\sum_{k=\infty}^{+\infty} 2\pi a_k \delta(\omega-k\omega_0), \qquad (4.22)$$

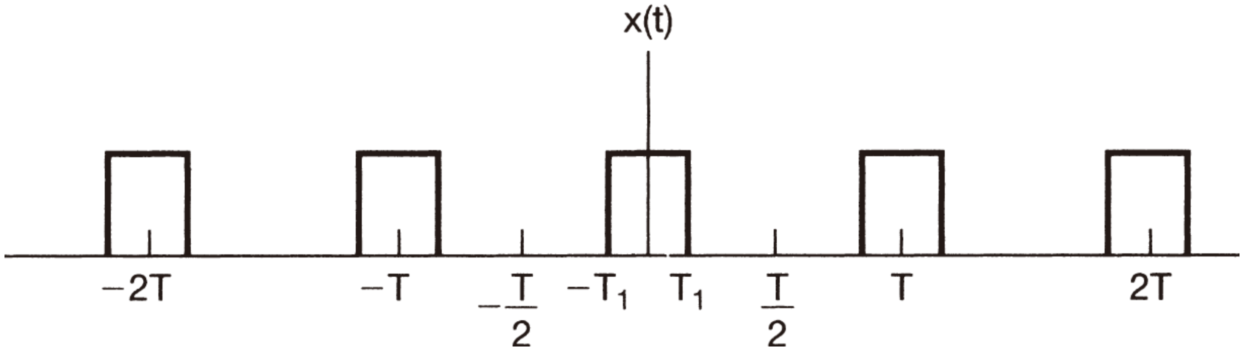
then the application of eq. (4.8) yields

$$x(t)=\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \qquad (4.23)$$

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega-k\omega_0)$$



Example 4.6



$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\omega - k\omega_0)$$

Consider again the square wave illustrated in Figure 4.1. The Fourier series coefficients for this signal are

$$a_k = \frac{\sin k\omega_0 T_1}{\pi k},$$

and the Fourier transform of the signal is

$$X(j\omega) = \sum_{k=-\infty}^{+\infty} \frac{2\sin k\omega_0 T_1}{k}\delta(\omega - k\omega_0),$$

Example 4.6

$$X(j\omega)=\sum_{k=-\infty}^{+\infty}2\omega_0 T_1\frac{\sin k\omega_0 T_1}{k\omega_0 T_1}\delta(\omega-k\omega_0),\omega_0=\frac{2\pi}{T}$$

$$a_k=\frac{\sin k\omega_0 T_1}{\pi k}=\frac{\omega_0 T_1}{\pi}\frac{\sin k\omega_0 T_1}{\pi k\omega_0 T_1},\quad =\sum_{k=-\infty}^{+\infty}\pi\frac{\sin k\pi/2}{k\pi/2}\delta(\omega-k\omega_0)$$

which is sketched in figure 4.12 for $T=4T_1$. In comparison with Figure 3.7 (a), the only differences are a **proportionality factor of 2*π*** and the use of **impulses** rather than a bar graph.

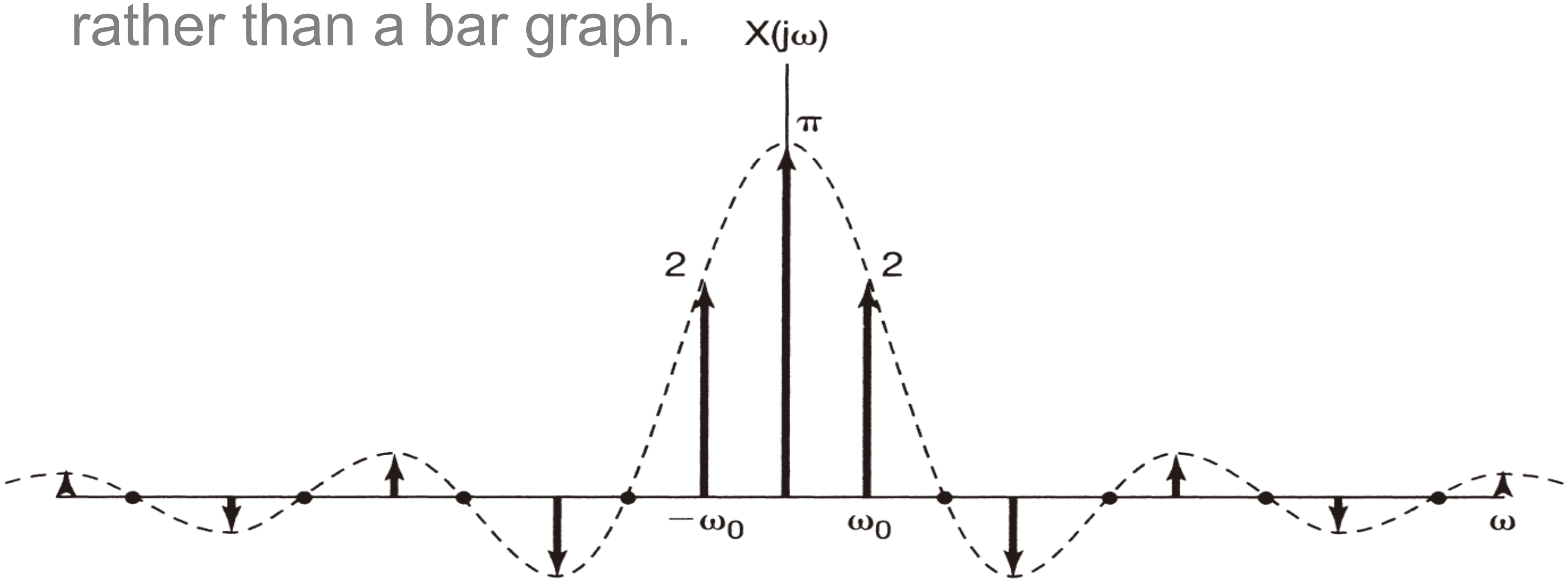


Figure 4.12 Fourier transform of a symmetric periodic square wave.

4.3 Properties of the Continuous-time Fourier Transform

As developed in Section 4.1, a signal *x(t)* and its Fourier transform $X(j\omega)$ are related by the Fourier transform synthesis and analysis equations,

[eq. (4.8)]

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}\,d\omega \qquad (4.24)$$

and

[eq. (4.9)]

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}\,dt. \qquad (4.25)$$

4.3 Properties of the Continuous-time Fourier Transform

refer to *x(t)* and $X(j\omega)$ as a Fourier transform pair with the notation

$$x(t) \xleftrightarrow{F} X(j\omega).$$

Thus, with reference to Example 4.1,

$$\frac{1}{a+j\omega} = F\left\{e^{-at}u(t)\right\},$$

$$e^{-at}u(t) = F^{-1}\left\{\frac{1}{a+j\omega}\right\},$$

and

$$e^{-at}u(t) \xleftrightarrow{F} \frac{1}{a+j\omega}.$$

4.3.1 Linearity

If

$$x(t) \xleftrightarrow{F} X(j\omega)$$

and

$$y(t) \xleftrightarrow{F} Y(j\omega),$$

Then

$$ax(t) + by(t) \xleftrightarrow{F} aX(j\omega) + bY(j\omega). \quad (4.26)$$

線性性質(重疊性質適用)

4.3.2 Time Shifting

If

$$x(t) \xleftrightarrow{F} X(j\omega),$$

Then

時間移位性質 $x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega).$ (4.27)

To establish this property, consider eq. (4.24);

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\, d\omega.$$

4.3.2 Time Shifting $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega.$

Replacing *t* by $t - t_0$ in this equation, we obtain

$$x(t - t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}\,d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(e^{-j\omega t_0}X(j\omega)\right)e^{j\omega t}\,d\omega.$$

Recognizing this as the synthesis equation for $x(t - t_0)$, we conclude that

$$F\{x(t - t_0)\} = e^{-j\omega t_0}X(j\omega).$$

4.3.2 Time Shifting

if we express *X(jω)* in polar form as

$$F\{x(t)\} = X(j\omega) = |X(j\omega)| e^{j\measuredangle X(j\omega)},$$

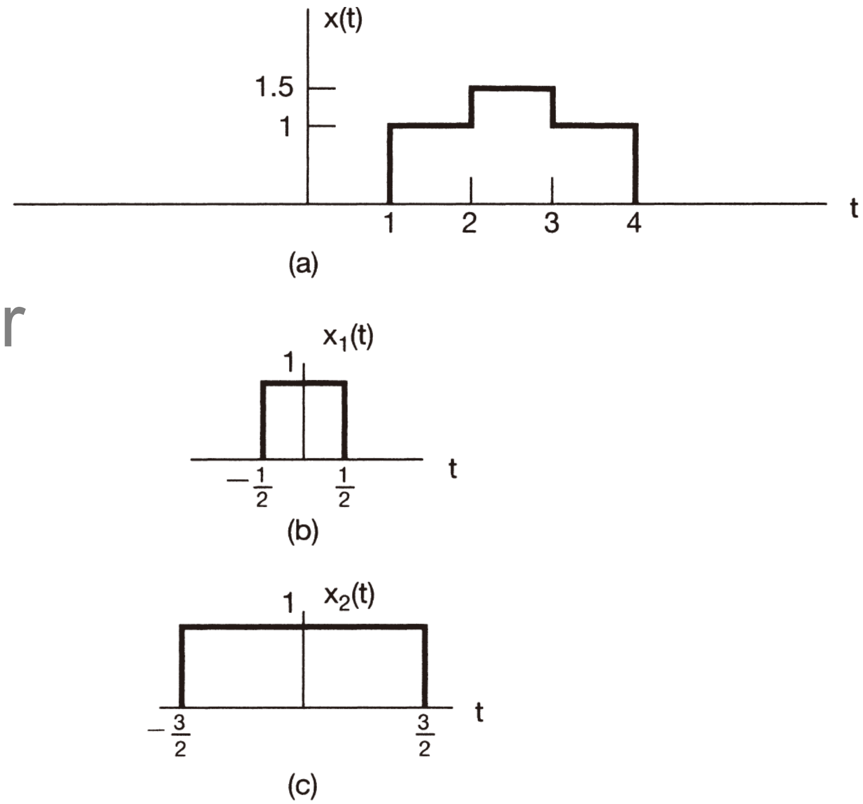
then

$$F\{x(t-t_0)\} = e^{-j\omega t_0} X(j\omega) = |X(j\omega)| e^{j[\measuredangle X(j\omega) - \omega t_0]}.$$

a time shift results to a phase shift in FT

Example 4.9

To illustrate the usefulness of the Fourier transform linearity and time-shift properties, let us consider the evaluation of the Fourier transform of the signal *x(t)* shown in Figure 4.15(a).

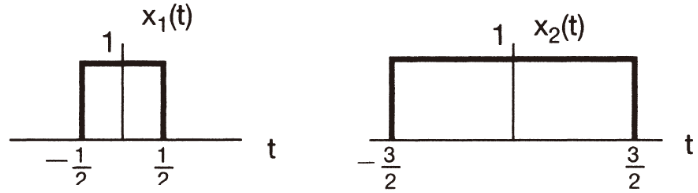


First, we observe that *x(t)* can be expressed as the linear combination

$$x(t) = \frac{1}{2} x_1(t - 2.5) + x_2(t - 2.5),$$

Example 4.9

$$X(j\omega) = 2\frac{\sin \omega T_1}{\omega}.$$



where the signals $x_1(t)$ and $x_2(t)$ are the rectangular plus signals shown in Figure 4.15(b) and (c). Then, using the result from Example 4.4, we obtain

$$X_1(j\omega) = \frac{2\sin(\omega/2)}{\omega} \qquad and \qquad X_2(j\omega) = \frac{2\sin(3\omega/2)}{\omega}.$$

Finally, using the linearity and time-shift properties of the Fourier transform yields

$$X(j\omega) = e^{-j5\omega/2}\left\{\frac{\sin(\omega/2) + 2\sin(3\omega/2)}{\omega}\right\}.$$

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5),\ F\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega).$$

4.3.3 Conjugation and Conjugate Symmetry

The conjugation property states that if

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega),$$

then

共軛性質 (4.28)

$$x^*(t) \overset{F}{\longleftrightarrow} X^*(-j\omega).$$

4.3.3 Conjugation and Conjugate Symmetry

$$X(j\omega)=\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt.(4.25)$$

This property follows from the evaluation of the complex conjugate eq. (4.25):

$$X^*(j\omega)=\left[\int_{-\infty}^{+\infty}x(t)e^{-j\omega t}dt\right]^*$$

$$=\int_{-\infty}^{+\infty}x^*(t)e^{j\omega t}dt.$$

Replacing ω by –ω, we see that

$$X^*(-j\omega)=\int_{-\infty}^{+\infty}x^*(t)e^{-j\omega t}dt. \qquad (4.29)$$

4.3.3 Conjugation and Conjugate Symmetry

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega) \qquad x^*(t) \overset{F}{\longleftrightarrow} X^*(-j\omega).$$

if $x(t)$ is real so that $x^*(t) = x(t)$,

we have

$$X^*(-j\omega) = X(j\omega),$$

and, by replacing ω and –ω, we have

$$X^*(j\omega) = X(-j\omega) \quad (4.30)$$

conjugate symmetry

4.3.3 Conjugation and Conjugate Symmetry

From Example 4.1, with a real signal $x(t) = e^{-at}u(t)$,

$$X(j\omega) = \frac{1}{a + j\omega} = \frac{a - j\omega}{a^2 + \omega^2}$$

and

$$X(-j\omega) = \frac{1}{a - j\omega} = \frac{a + j\omega}{a^2 + \omega^2} = X^*(j\omega).$$

4.3.3 Conjugation and Conjugate Symmetry

$$X^*(j\omega) = X(-j\omega) \quad (4.30)$$

As one consequence of eq. (4.30), if we express *X(jω)* in **rectangular** form as

$$X(j\omega) = \Re e\{X(j\omega)\} + j\Im m\{X(j\omega)\},$$

Then if *x(t)* is real,

and $$\Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \qquad \textit{even}$$

$$\Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\}. \qquad \textit{odd}$$

4.3.3 Conjugation and Conjugate Symmetry

$$X^*(j\omega) = X(-j\omega) \quad (4.30)$$

if we express *X(jω)* in **polar** form as

$$X(j\omega) = |X(j\omega)| e^{j\measuredangle X(j\omega)},$$

As a further consequence of eq. (4.30), if *x(t)* is both **real** and **even**, then *X(jω)* will also be **real** and **even**. To see this, we write

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(t) e^{j\omega t} dt,$$

$$\text{since} \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt.$$

4.3.3 Conjugation and Conjugate Symmetry

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.$$

or, with the substitution *τ = -t,*

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(-\tau)e^{-j\omega\tau}d\tau.$$

since *x(-τ) = x(τ) (even),* we have

$$X(-j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau$$

$$= X(j\omega).$$

$$X^{*}(j\omega) = X(-j\omega) \qquad (4.30)$$

if *x(t)* is both **real** and **odd**, then *X(jω)* will also be **imaginary** and **odd**.

4.3.3 Conjugation and Conjugate Symmetry

Finally, as was discussed in Chapter 1, a **real** function *x(t)* can always be expressed in terms of the sum of an even function $x_e(t)$ $=\mathcal{E}v\{x(t)\}$ and an odd function $x_0(t)=\mathcal{O}d\{x(t)\}$; that is,

$$x(t) = x_e(t) + x_0(t).$$

From the linearity of the Fourier transform,

$$F\{x(t)\} = F\{x_e(t)\} + F\{x_0(t)\},$$

4.3.3 Conjugation and Conjugate Symmetry

and from the preceding discussion, $F\{x_e(t)\}$ is a real function and $F\{x_0(t)\}$ is purely imaginary. Thus, we can conclude that, with *x(t)* real,

$$x(t) \xleftrightarrow{F} X(j\omega),$$

$$\varepsilon v\{x(t)\} \xleftrightarrow{F} \Re e\{X(j\omega)\},$$

$$\sigma d\{x(t)\} \xleftrightarrow{F} j\Im m\{X(j\omega)\}.$$

訊號的偶函數部份的傅立葉轉換等於原訊號傅立葉轉換的實部。訊號的奇函數部份的傅立葉轉換等於原訊號傅立葉轉換的虛部。

Example 4.10

Consider again the Fourier transform evaluation of Example 4.2 for the signal $x(t) = e^{-a|t|}$, where a $>$ 0. This time we will utilize the symmetry properties of the Fourier transform to aid the evaluation process.

From Example 4.1, we have

$$e^{-at}u(t) \stackrel{F}{\longleftrightarrow} \frac{1}{a + j\omega}.$$

Example 4.10

Note that for $t > 0$, $x(t)$ equals $e^{-at}u(t)$, while for $t < 0$, $x(t)$ takes on mirror image values. That is,

$$x(t) = e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t)$$

$$= 2\left[\frac{e^{-at}u(t) + e^{at}u(-t)}{2}\right]$$

$$= 2\varepsilon v\left\{e^{-at}u(t)\right\}.$$

Example 4.10

$$x(t) \xleftrightarrow{F} X(j\omega),$$
$$\mathcal{E}v\{x(t)\} \xleftrightarrow{F} \Re e\{X(j\omega)\},$$
$$\mathcal{O}d\{x(t)\} \xleftrightarrow{F} j\Im m\{X(j\omega)\}.$$

Since e^{-at} u(t) is **real** valued, the symmetry properties of the Fourier transform lead us the conclude that

$$\mathcal{E}v\{e^{-at}u(t)\} \xleftrightarrow{F} \Re e\left\{\frac{1}{a+j\omega}\right\}.$$

It follows that $x(t) = 2\mathcal{E}v\{e^{-at}u(t)\}.$

$$X(j\omega) = 2\Re e\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2},$$

which is the same as the answer found in Example 4.2.

4.3.4 Differentiation and Integration

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \quad (4.24)$$

By differentiation both sides of the Fourier transform synthesis equation (4.24), we obtain

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X(j\omega)e^{j\omega t}d\omega.$$

Therefore,

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega). \quad (4.31)$$

This is **very useful for LTI described by differential equations**

訊號微分性質時域的微分相對於頻域乘以$j\omega$。

4.3.4 Differentiation and Integration

$$one \quad might \quad guess \ \int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega).$$

The precise relationship is

$$\int_{-\infty}^{t} x(\tau)\,d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega).$$

(4.32)

訊號積分性質

$$dc \quad (average \quad value): \ X(0) = \int_{-\infty}^{+\infty} x(t)\,dt.$$

Example 4.11

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega).$$

Let us determine the Fourier transform *X(jω)* of the unit step *x(t) = u(t),* making use of eq.(4.32) and the knowledge that

$$g(t) = \delta(t) \xleftrightarrow{F} G(j\omega) = 1.$$

Noting that

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau$$

and taking the Fourier transform of both sides, we obtain

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega),$$

Example 4.11

Where we have used the integration property listed in Table 4.1. Since $G(j\omega) = 1$, we conclude that

$$X(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega).$$

Observe that we can apply the differentiation property of eq. (4.31) to recover the transform of the impulse. That is,

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{F} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1,$$

where the last equality follows from the fact that $\omega\delta(\omega)=0$.

4.3.5 Time and Frequency Scaling

If

$$x(t) \overset{F}{\longleftrightarrow} X(j\omega),$$

Then

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right),$$

時間刻度變換 (4.34)

where a is a nonzero real number. This property follows directly from the definition of the Fourier transform—specifically,

$$F\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt.$$

4.3.5 Time and Frequency Scaling

Using the substitution τ=at , we obtain

$$F\{x(at)\} = \begin{cases} \frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a\rangle 0 \\ -\frac{1}{a}\int_{-\infty}^{+\infty} x(\tau)e^{-j(\omega/a)\tau}d\tau, & a\langle 0 \end{cases}$$

Thus, aside from the amplitude factor 1/lal, a linear scaling in time by a factor of corresponds to a linear scaling in frequency by a factor of 1/a, and vice versa. Also, letting a = -1

$$x(-t) \xleftrightarrow{F} X(-j\omega).$$

(4.35)

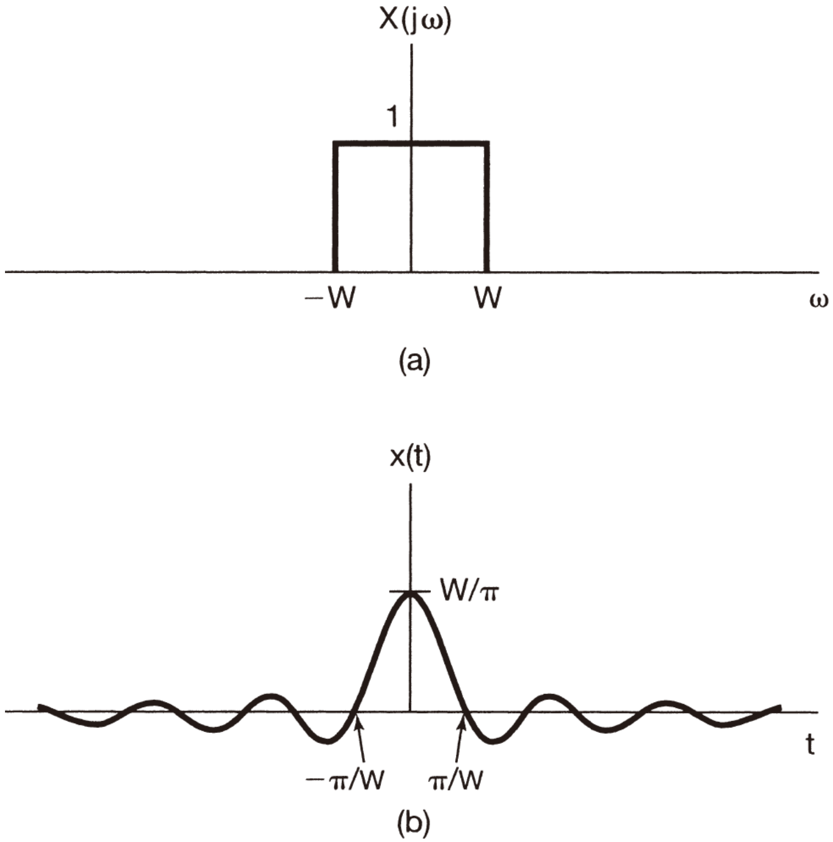
訊號時間倒轉對應的傅立葉轉換為頻率倒轉。

Example 4.5
(Inverse Relationship)

$$x(at) \overset{F}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right),$$

Consider the signal *x(t)* whose Fourier transform is

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} .(4.18)$$



Using the synthesis equation (4.8), we can determine

$$x(t) = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega t} d\omega = \frac{\sin Wt}{\pi t} .(4.19)$$

4.3.6 Duality

The symmetry between

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega \qquad (4.24)$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt. \qquad (4.25)$$

4.3.6 Duality

In the former example we derived the Fourier transform pair

$$x_1(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & |t| > T_1 \end{cases} \xleftrightarrow{F} X_1(j\omega) = \frac{2\sin\omega T_1}{\omega}, \quad (4.36)$$

脈波型式(時域) ⟷ sin函數型式(頻域)

while in the latter we considered the pair

$$x_2(t) = \frac{\sin Wt}{\pi t} \xleftrightarrow{F} X_2(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases}. \quad (4.37)$$

sin函數型式(時域) ⟷ 脈波型式(頻域)

4.3.6 Duality

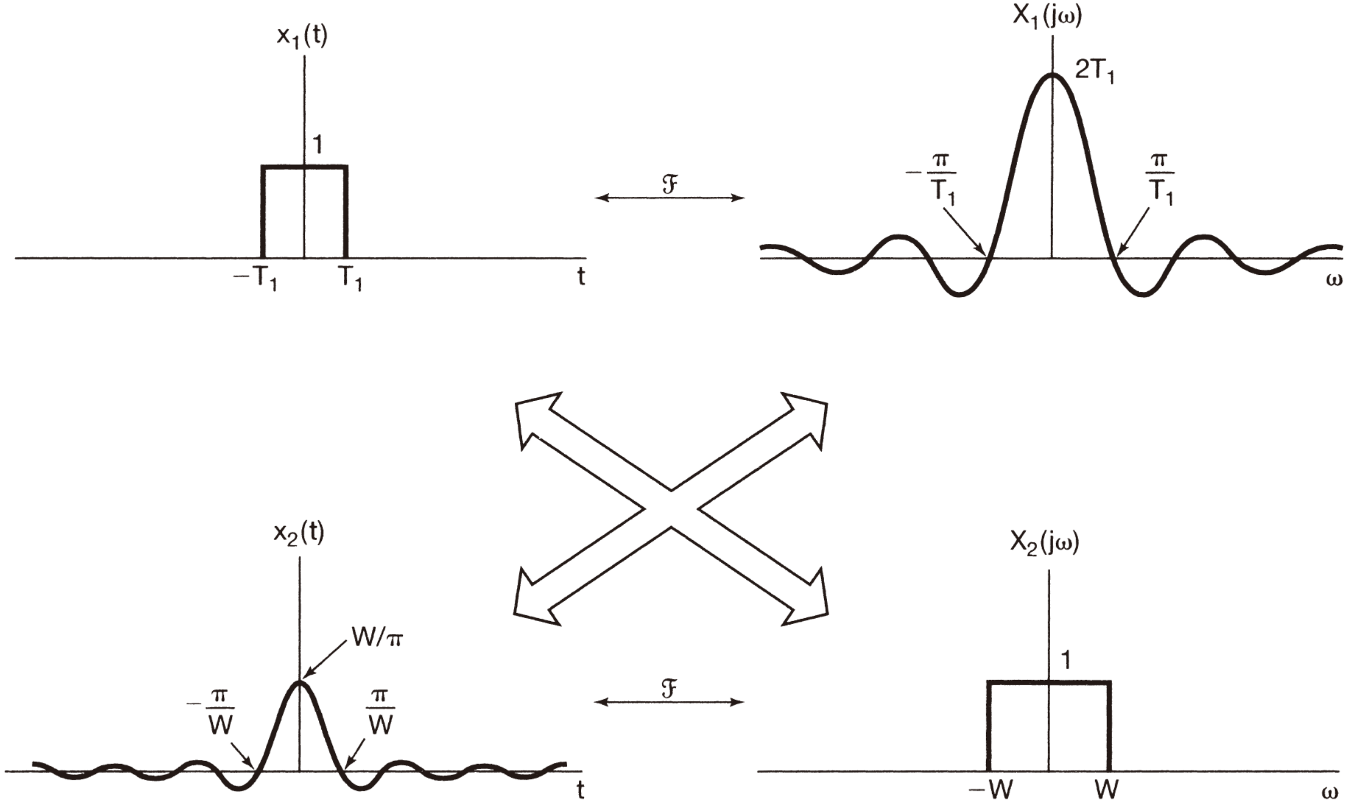


Figure 4.17 Relationship between the Fourier transform pairs of eqs. (4.36) and (4.37).

Example 4.13

Let us consider using duality to find the Fourier transform *G(jω)* of the signal

$$g(t) = \frac{2}{1+t^2}.$$

In Example 4.2 we encountered a Fourier transform pair in which the Fourier transform, as a function of *ω*, had a form similar to that of the signal *x(t).* Specifically, suppose we consider a signal *x(t)* whose Fourier transform is

$$X(j\omega) = \frac{2}{1+\omega^2}.$$

Then, from Example 4.2,

$$x(t) = e^{-|t|} \stackrel{F}{\longleftrightarrow} X(j\omega) = \frac{2}{1+\omega^2}.$$

Example 4.13

The synthesis equation for this Fourier transform pair is

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{j\omega t}\,d\omega.$$

Multiplying this equation by 2π and replacing t by $-t$, we obtain

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty}\left(\frac{2}{1+\omega^2}\right)e^{-j\omega t}\,d\omega.$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.$$

Example 4.13

$$2\pi e^{-|t|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+\omega^2}\right) e^{-j\omega t} d\omega.$$

Now, interchanging the names of the variables t and ω, we find that

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \left(\frac{2}{1+t^2}\right) e^{-j\omega t} dt. \tag{4.38}$$

The right-hand side of eq. (4.38) is the Fourier transform analysis equation for $2/(1+ t^2)$, and thus, we conclude that

$$G(\omega) = F\left\{\frac{2}{1+t^2}\right\} = 2\pi e^{-|\omega|}. \qquad g(t) = \frac{2}{1+t^2}.$$

4.3.6 Duality

$$\frac{dx(t)}{dt} \xleftrightarrow{F} j\omega X(j\omega). \qquad X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.$$

To determine the precise form of this dual property, we can proceed in a fashion exactly analogous to that used in Section 4.3.4. Thus, if we differentiate the analysis equation (4.25) with respect to ω, we obtain

That is,
$$\frac{dX(j\omega)}{d\omega} = \int_{-\infty}^{+\infty} -jtx(t)e^{-j\omega t}dt. \tag{4.39}$$

$$-jtx(t) \xleftrightarrow{F} \frac{dX(j\omega)}{d\omega}. \tag{4.40}$$

由微分性質及對偶性質而得。

4.3.6 Duality

$$x(t-t_0) \xleftrightarrow{F} e^{-j\omega t_0} X(j\omega).(4.27)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega} X(j\omega) + \pi X(0)\delta(\omega).(4.32)$$

Similarly, we can derive the dual properties of eqs. (4.27) and (4.32):

$$e^{j\omega_0 t} x(t) \xleftrightarrow{F} X\left(j(\omega-\omega_0)\right) \qquad (4.41)$$

由時間移位性質及對偶性質而得。

and

$$-\frac{1}{jt} x(t) + \pi x(0)\delta(t) \xleftrightarrow{F} \int_{-\omega}^{\omega} x(\eta)d\eta. \qquad (4.42)$$

由積分性質及對偶性質而得。

4.3.7 Parseval's Relation

$$x^*(t) \xleftrightarrow{F} X^*(-j\omega). \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

If *x(t)* and *X(jω)* are a Fourier transform pair, then

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega. \tag{4.43}$$

巴斯瓦關係式(定理)

This expression, referred to as Parseval's relation, follows from direct application of the Fourier transform. Specifically,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)e^{-j\omega t}d\omega\right] dt.$$

4.3.7 Parseval's Relation

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt.$$

Reversing the order of integration gives

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]d\omega.$$

The bracketed term is simply the Fourier transform of *x(t);* thus,

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega.$$

$|X(j\omega)|^2$ often refer to as **energy density spectrum**.

此定理意指訊號*x(t)*的總能量等於$|X(j\omega)|^2/2\pi$對整個頻率軸積分。故$|X(j\omega)|^2$常稱為「能量密度頻譜」。

Example 4.14

For each of the Fourier transforms shown in Figure 4.18, we wish to evaluate the following time-domain expressions:

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$D = \frac{d}{dt} x(t)\Big|_{t=0}$$

To evaluate E in the frequency domain, we may use Parseval's relation. That is,

$$(4.44) \quad E = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(j\omega)|^2 \, d\omega$$

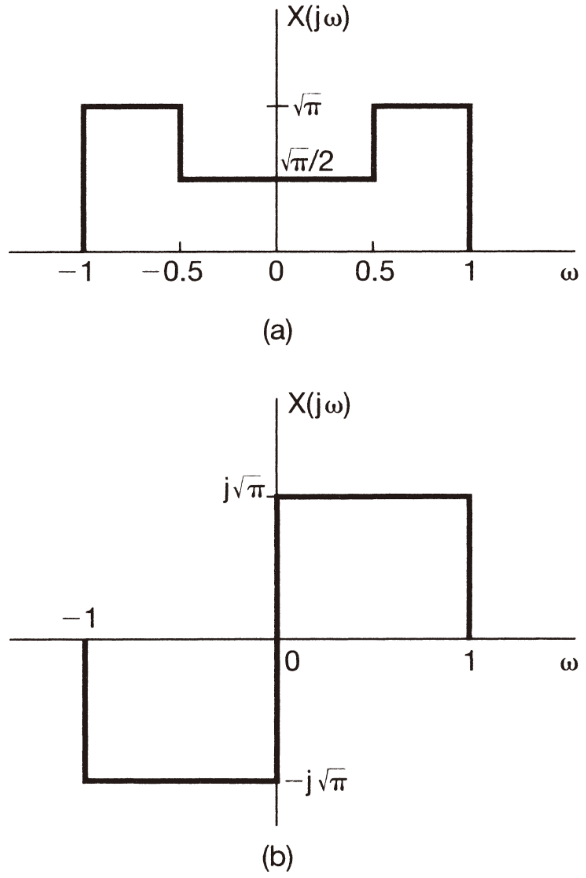


Figure 4.18 The Fourier transforms considered in Example 4.14.

Example 4.14

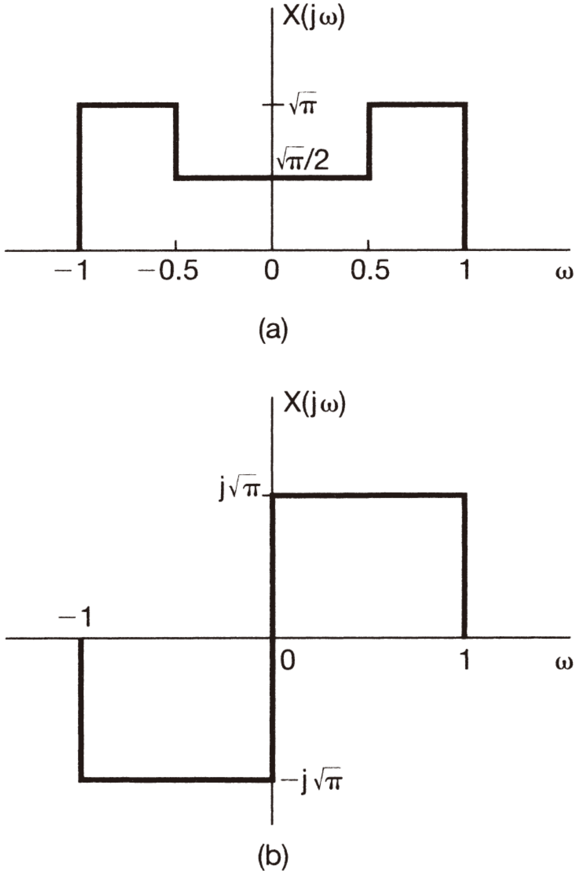


Figure 4.18 The Fourier transforms considered in Example 4.14.

$$E = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left|X(j\omega)\right|^2 d\omega$$

$$E = \frac{1}{2\pi}\left(\frac{\pi}{4} + \pi\right) = \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

$$E = \frac{1}{2\pi}(\pi + \pi) = 1$$

which evaluates to for Figure 4.18(a) and to 1 for Figure 4.18(b)

Example 4.14

$$g(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G(j\omega)e^{j\omega t}d\omega$$

To evaluate D in the frequency domain, we first use the differentiation property to observe that

$$g(t) = \frac{d}{dt}x(t) \overset{F}{\longleftrightarrow} j\omega X(j\omega) = G(j\omega).$$

Noting that

$$D = g(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G(j\omega)d\omega \tag{4.45}$$

we conclude:

$$D = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)d\omega \tag{4.46}$$

Example 4.14

$$D = \frac{1}{2\pi}\int_{-\infty}^{\infty} j\omega X(j\omega)\,d\omega$$

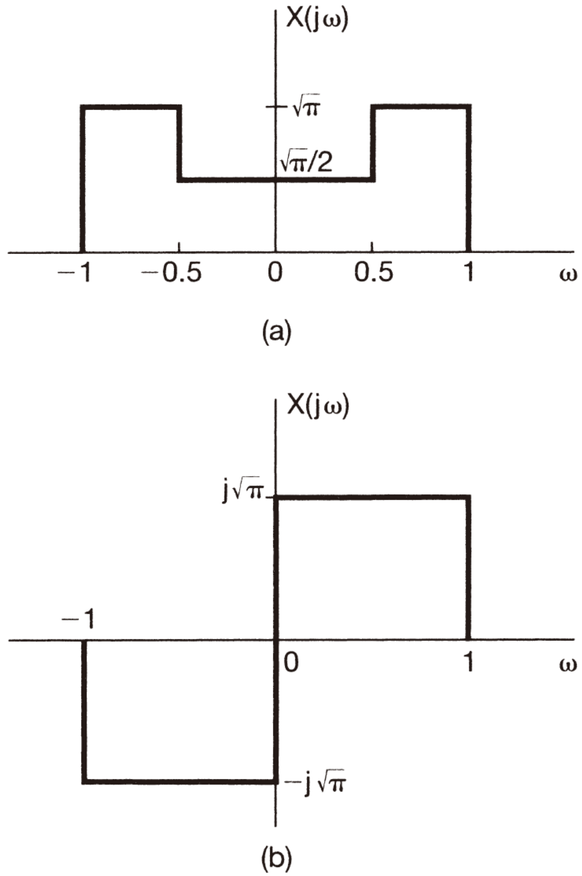


Figure 4.18 The Fourier transforms considered in Example 4.14.

$$D = 0$$

$$D = \frac{1}{2\pi}(-0.5\sqrt{\pi} + -0.5\sqrt{\pi}) = \frac{-\sqrt{\pi}}{2\pi} = \frac{-1}{2\sqrt{\pi}}$$

which evaluates to 0 for Figure 4.18(a) and to $\frac{-1}{(2\sqrt{\pi})}$ for Figure 4.18(b)

4.4 The Convolution Property

referring back to eq. (4.7), $x(t)$ is expressed as the limit of a sum; that is,

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega = \lim_{\omega_0 \to 0} \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0. \tag{4.47}$$

As developed in Sections 3.2 and 3.8, the response of a linear system with impulse response $h(t)$ to a complex exponential $e^{jk\omega_0 t}$ is $H(jk\omega_0)e^{jk\omega_0 t}$, where

$$H(jk\omega_0) = \int_{-\infty}^{+\infty} h(t)e^{-jk\omega_0 t}dt. \tag{4.48}$$

4.4 The Convolution Property

From superposition [see eq.(3.124)], we then have

$$\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)e^{jk\omega_0 t}\omega_0 \rightarrow \frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0,$$

and thus, from eq. (4.47), the response of the linear system to $x(t)$ is

$$y(t) = \lim_{\omega_0 \to 0}\frac{1}{2\pi}\sum_{k=-\infty}^{+\infty} X(jk\omega_0)H(jk\omega_0)e^{jk\omega_0 t}\omega_0$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)H(j\omega)e^{j\omega t}d\omega. \qquad (4.49)$$

4.4 The Convolution Property

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(j\omega) H(j\omega) e^{j\omega t} d\omega.$$

Since *y(t)* and its Fourier transform *Y(jω)* are related by

$$y(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} Y(j\omega) e^{j\omega t} d\omega, \qquad (4.50)$$

we can identify *Y(jω)* from eq. (4.49), yielding

$$Y(j\omega) = X(j\omega) H(j\omega). \qquad (4.51)$$

As a more formal derivation, we consider the convolution integral

$$y(t) = \int_{-\infty}^{+\infty} x(\tau) h(t-\tau) d\tau. \qquad (4.52)$$

4.4 The Convolution Property

We desire *Y(jω)*, which is

$$Y(j\omega) = F\{y(t)\} = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt. \qquad (4.53)$$

Interchanging the order of integration and noting that x(τ) does not depend on t, we have

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau. \qquad (4.54)$$

4.4 The Convolution Property

$$H(\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt; e^{-j\omega\tau}H(\omega) = \int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt.$$

By the time-shift property, eq. (4.27), the bracketed term is $e^{-j\omega\tau}H(j\omega)$. Substituting this into eq. (4.54) yields

$$Y(j\omega) = \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau.$$

The integral is $X(j\omega)$, and hence,

$$Y(j\omega) = H(j\omega)X(j\omega).$$

That is,

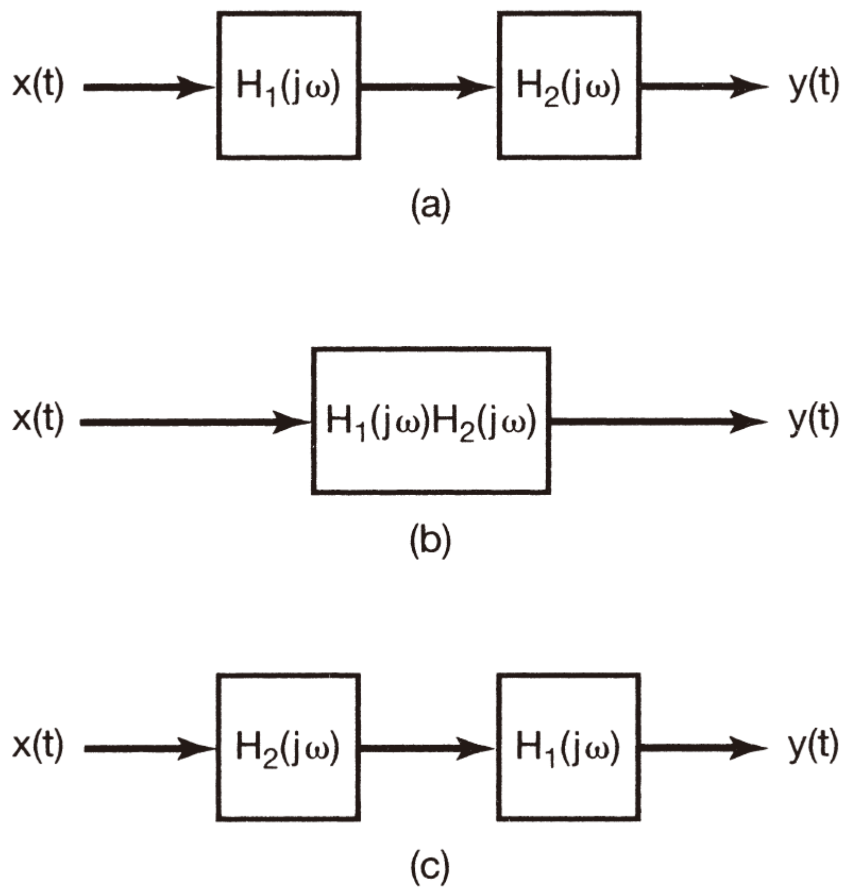
$$y(t) = h(t) * x(t) \stackrel{F}{\longleftrightarrow} Y(j\omega) = H(j\omega)X(j\omega). \quad (4.56)$$

連續時間傅立葉轉換的迴旋運算性質

上式在訊號與系統分析上是極為重要的，它將時域中較複雜的迴旋運算轉換至頻域中較簡單的乘法。

4.4 The Convolution Property

As illustrated in Figure 4.19, since the impulse response of the cascade of two LTI systems is the convolution of the individual impulse responses, the convolution property then implies that the overall frequency response of the cascade of two systems is simply the product of the individual frequency responses.



4.4 The Convolution Property

(Sec. 4.12) convergence of FT is guaranteed only under certain condition. Hence, frequency response cannot be defined for every LTI system.

If, however, an LTI system is **stable**, then, as we saw in Section 2.3.7 and Problem 2.49, its impulse response is absolutely integrable; that is,

$$\int_{-\infty}^{+\infty} |h(t)| dt < \infty.$$

This is one of Dirichlet condition. Assuming other 2 conditions are satisfied, it will have a valid H(jω)

4.4 The Convolution Property

In using Fourier analysis to study LTI systems, we will be restricting ourselves to systems whose **impulse responses possess Fourier transforms**.

In Chap. 9, we will develop a generalization of CTFT, the **Laplace transform**, for examine unstable LTI system.

為了利用轉換法來檢視不穩定的LTI系統，必須藉助一種傅立葉轉換的一般化型式，即「拉氏轉換」。

Example 4.15

Consider a continuous-time LTI system with impulse response

$$h(t) = \delta(t - t_0). \tag{4.58}$$

The frequency response of this system is the Fourier transform of *h(t)* and is given by

$$H(j\omega) = e^{-j\omega t_0}. \tag{4.59}$$

Thus, for any input *x(t)* with Fourier transform *X(jω)*, the Fourier transform of the output is

$$\begin{aligned} Y(j\omega) &= H(j\omega)X(j\omega) \\ &= e^{-j\omega t_0}\, X(j\omega). \end{aligned} \tag{4.60}$$

Example 4.15

This result, in fact, is consistent with the time-shift property of Section 4.3.2. Specifically, a system for which the impulse response is $\delta(t-t_0)$ applies a time shift of t_0 to the input—that is,

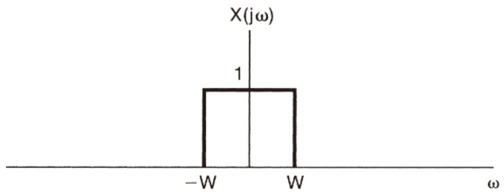
$$y(t) = x(t-t_0).$$

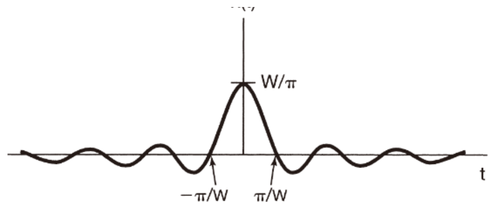
Example 4.15

Thus, the shifting property given in eq. (4.27) also yields eq. (4.60). Note that, either from our discussion in Section 4.3.2 or directly from eq. (4.59), the frequency response of a system that is a pure time shift has unity magnitude at all frequencies (i.e., $\left|e^{-j\omega t_0}\right| = 1$) and has a phase characteristic $-\omega t_0$ that is a linear function of ω.

$$
\begin{aligned}
Y(j\omega) &= H(j\omega)X(j\omega) \\
&= e^{-j\omega t_0} X(j\omega).
\end{aligned}
$$

Example 4.20





As another illustration of the usefulness of the convolution property, let us consider the problem of determining the response of an ideal lowpass filter to an input signal *x(t)* that has the form of a sinc function. That is,

$$x(t) = \frac{\sin \omega_i t}{\pi t}.$$

Of course, the impulse response of the ideal lowpass filter is of a similar form, namely,

$$h(t) = \frac{\sin \omega_c t}{\pi t}.$$

Example 4.20

The filter output *y(t)* will therefore be the convolution of two sinc functions, which, as we now show, also turns out to be a sinc function. A particularly convenient way of deriving this result is to first observe that

$$Y(j\omega) = X(j\omega)H(j\omega),$$

where

$$X(j\omega) = \begin{cases} 1 & |\omega| \le \omega_i \\ 0 & elsewhere \end{cases}$$

and

$$H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & elsewhere \end{cases}.$$

Example 4.20

Therefore,

$$Y(j\omega) = \begin{cases} 1 & |\omega| \leq \omega_0 \\ 0 & elsewhere \end{cases},$$

where ω_0 is the smaller of the two numbers ω_i and ω_c.
Finally, the inverse Fourier transform of *Y(jω)* is given by

$$y(t) = \begin{cases} \dfrac{\sin \omega_c t}{\pi t} & if \omega_c \leq \omega_i \\ \dfrac{\sin \omega_i t}{\pi t} & if \omega_i \leq \omega_c \end{cases}$$

That is, depending upon which of ω_c and ω_i is smaller, the output is equal to either *x(t)* or *h(t)*.

4.5 The multiplication Property

Because of **duality** between the time and frequency domains, we would expect a dual property also to hold (i.e., that multiplication in the time domain corresponds to convolution in the frequency domain). Specifically,

$$r(t) = s(t)p(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P\big(j(\omega-\theta)\big)d\theta \tag{4.70}$$

乘法性質：

時域中訊號相乘對應至頻域中為個別的傅立葉轉換的迴旋積分。

4.5 The multiplication Property

$$r(t) = s(t)p(t) \leftrightarrow R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P\big(j(\omega-\theta)\big)d\theta$$

Multiplication of one signal by another can be thought of as using **one signal to scale or *modulate*** the amplitude of the other, and consequently, the multiplication of two signals is often referred to as ***amplitude modulation***. For this reason, eq. (4.70) is sometimes referred to as the *modulation property*.

兩訊號相乘常稱之為「振幅調變」。故(4.70)式常稱為「調變性質」。

Example 4.23

Another illustration of the usefulness of the Fourier transform multiplication property is provided by the problem of determining the Fourier transform of the signal

$$x(t) = \frac{\sin(t)\sin(t/2)}{\pi t^2}.$$

The key here is to recognize *x(t)* as the product of two sinc functions:

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right).$$

Example 4.23

$$X(j\omega) = \begin{cases} 1, & |\omega| < W \\ 0, & |\omega| > W \end{cases} .(4.18) \longleftrightarrow \quad x(t) = \frac{\sin Wt}{\pi t}$$

Applying the multiplication property of the Fourier transform, we obtain

$$X(j\omega) = \frac{1}{2} F\left\{\frac{\sin(t)}{\pi t}\right\} * F\left\{\frac{\sin(t/2)}{\pi t}\right\}.$$

Noting that the Fourier transform of each sinc function is a rectangular pulse, we can proceed to convolve those pulses to obtain the function *X(jω)* displayed in Figure 4.25.

Example 4.23

$$X(j\omega) = \frac{1}{2} F\left\{\frac{\sin(t)}{\pi t}\right\} * F\left\{\frac{\sin(t/2)}{\pi t}\right\}.$$

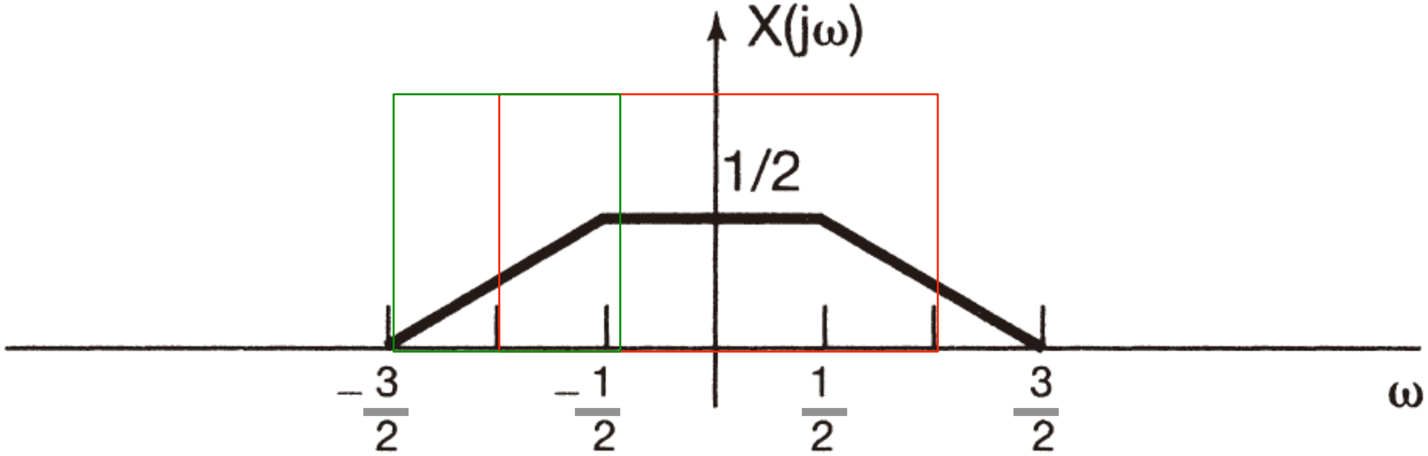


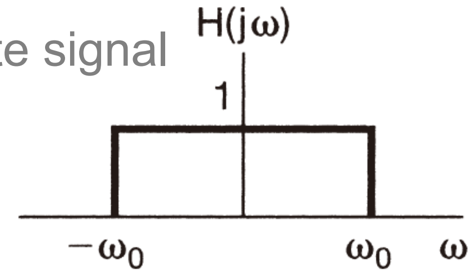
Figure 4.25 The Fourier transform of $x(t)$ in Example 4.23.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

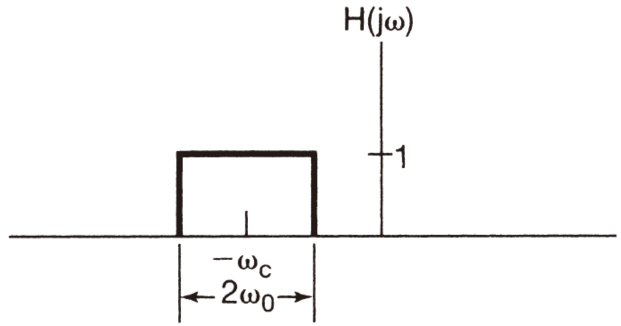
Multiplication property is important for **amplitude modulation** in communication system:

In a frequency-selective bandpass filter built with elements such as resistors, operational amplifiers, and capacitors, the center frequency depends on a number of element values, **all of which must be varied simultaneously** in the correct way if the center frequency is to be adjusted directly.

Easy to modulate signal with ω_c



Hard to change ω_c



4.5.1 Frequency-Selective Filtering with Variable Center Frequency

$$e^{j\omega_c t}x(t) \xleftrightarrow{F} X\left(j(\omega-\omega_c)\right)$$

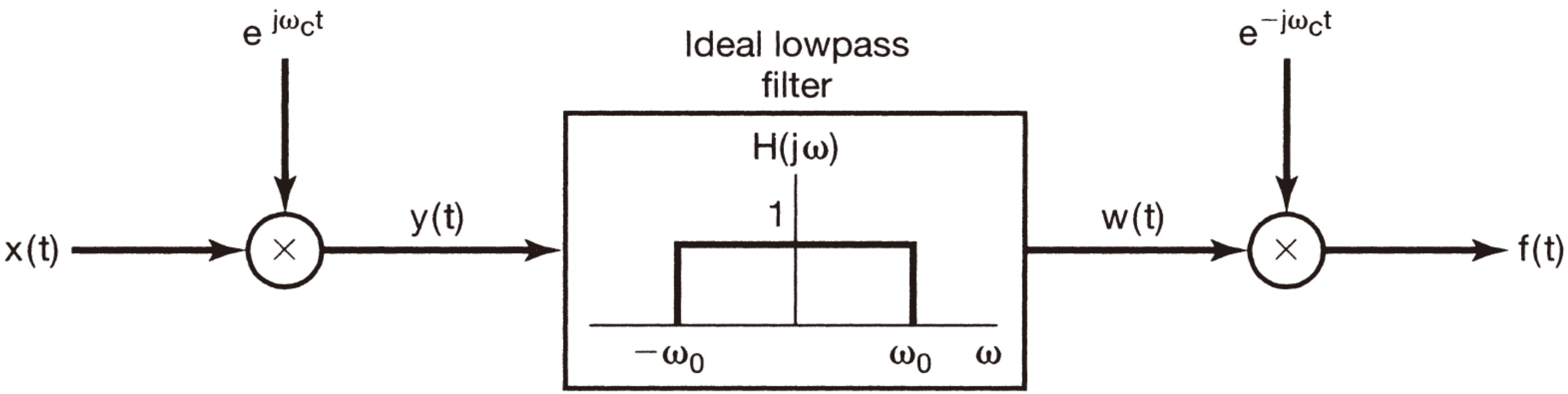
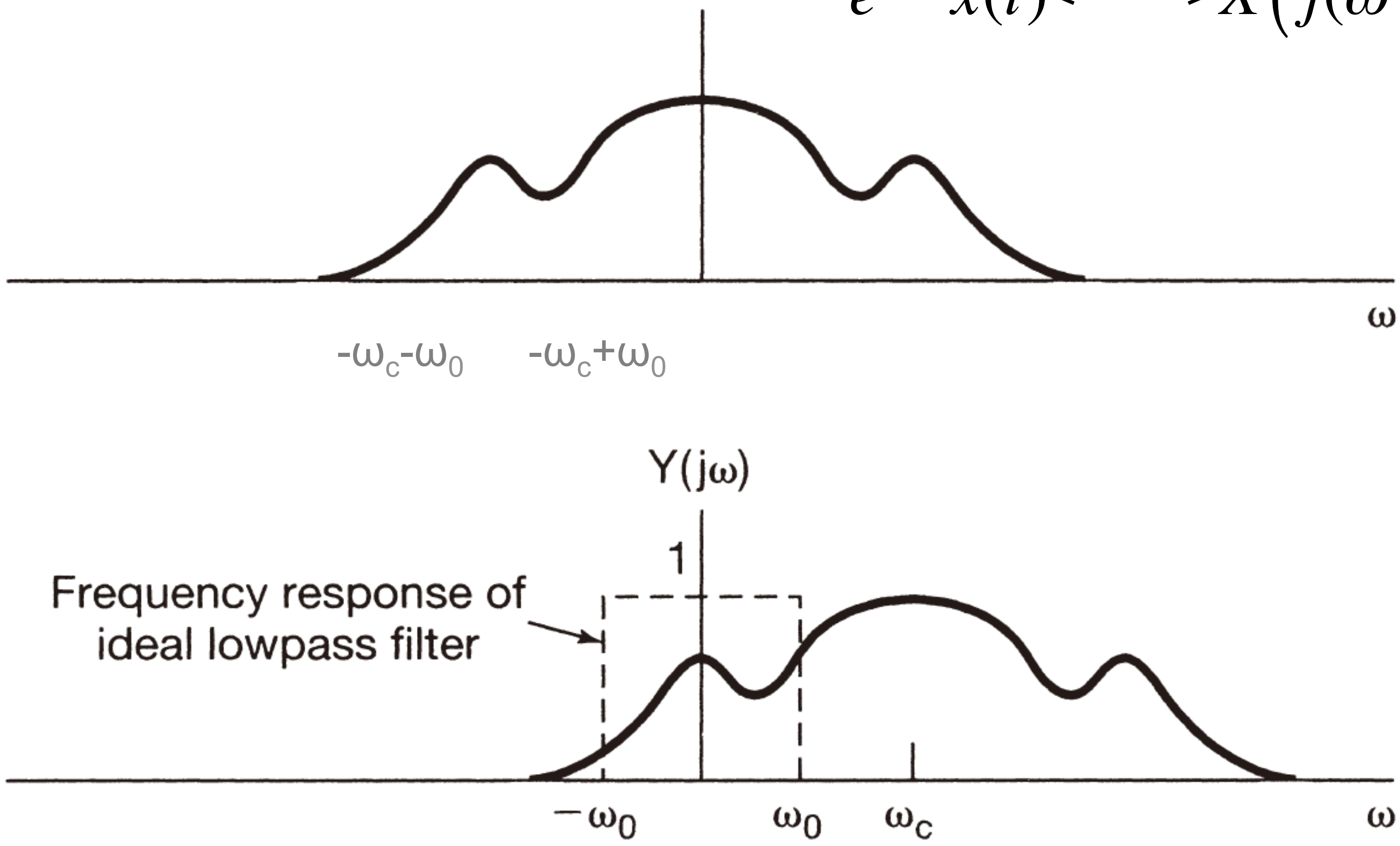


Figure 4.26 Implementation of a bandpass filter using amplitude modulation with a complex exponential carrier.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

$$e^{j\omega_c t}x(t) \xleftrightarrow{F} X\left(j(\omega-\omega_c)\right)$$



4.5.1 Frequency-Selective Filtering with Variable Center Frequency

$$e^{-j\omega_c t} w(t) \xleftrightarrow{F} F\left(j(\omega+\omega_c)\right)$$

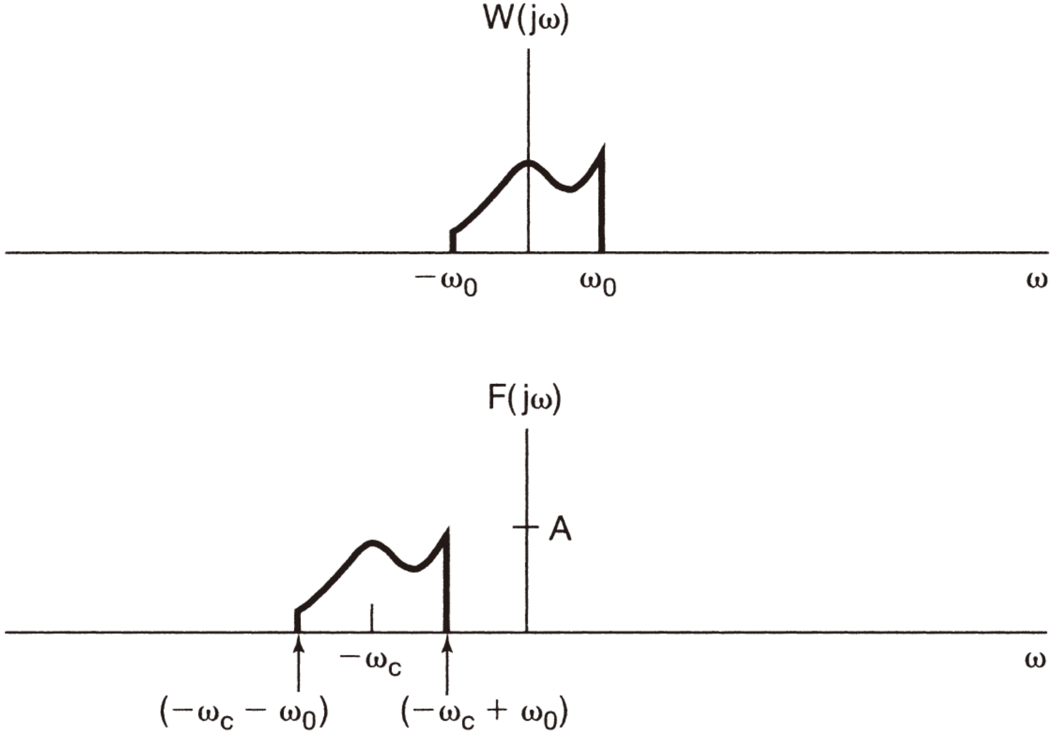


Figure 4.27 Spectra of the signals in the system of Figure 4.26.

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

$$e^{j\omega_0 t}x(t) \xleftrightarrow{F} X\left(j(\omega-\omega_0)\right)$$

The Fourier transform of $y(t)=e^{j\omega_c t}x(t)$ is

$$Y(j\omega)=\delta(j(\omega-\omega_c))*X(j\omega)=$$

$$\int_{-\infty}^{+\infty}\delta(j(\theta-\omega_c))X(j(\omega-\theta))d\theta = X(j(\omega-\omega_c))$$

the Fourier transform of $f(t)=e^{-jw_c t}w(t)$ is

$$F(j\omega)=\delta(j(\omega+\omega_c)*W(j\omega)=$$

$$\int_{-\infty}^{+\infty}\delta(j(\theta+\omega_c))W(j(\omega-\theta))d\theta = W\left(j(\omega+\omega_c)\right),$$

4.5.1 Frequency-Selective Filtering with Variable Center Frequency

So that the Fourier transform of $F(j\omega)$ is $W(j\omega)$ shifted to left by ω_c . From Figure 4.27, we observe that the overall system of Figure 4.26 is equivalent to an ideal bandpass filter with center frequency $-\omega_c$ and bandwidth $2\omega_0$, as illustrated in Figure 4.28.

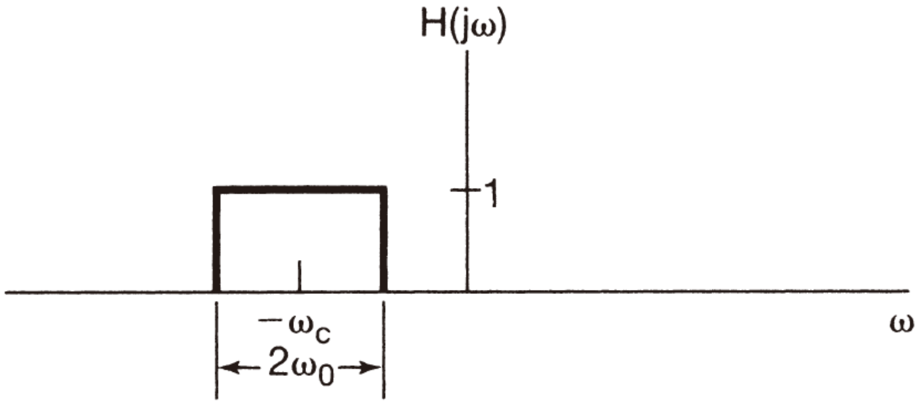


Figure 4.28 Bandpass filter equivalent of Figure 4.26.

4.6 Tables of Fourier Proerties and of Basic Fourier Transform Pairs

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM 表 4.1 傅立葉轉換的重要性質

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t)+by(t)$	$aX(j\omega)+bY(j\omega)$
4.3.2	Time Shifting	$x(t-t_0)$	$e^{-j\omega t_0}X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t}x(t)$	$X(j(\omega-\omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\lvert a\rvert}X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t)*y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty}X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t}x(t)dt$	$\frac{1}{j\omega}X(j\omega)+\pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases}X(j\omega)=X^*(-j\omega)\\ \mathcal{Re}\{X(j\omega)\}=\mathcal{Re}\{X(-j\omega)\}\\ \mathcal{Im}\{X(j\omega)\}=-\mathcal{Im}\{X(-j\omega)\}\\ \lvert X(j\omega)\rvert=\lvert X(-j\omega)\rvert\\ \sphericalangle X(j\omega)=-\sphericalangle X(-j\omega)\end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t)=\mathcal{Ev}\{x(t)\}$ [$x(t)$ real] $x_o(t)=\mathcal{Od}\{x(t)\}$ [$x(t)$ real]	$\mathcal{Re}\{X(j\omega)\}$ $j\mathcal{Im}\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty}|x(t)|^2dt=\frac{1}{2\pi}\int_{-\infty}^{+\infty}|X(j\omega)|^2d\omega$$

4.6 Tables of Fourier Proerties and of Basic Fourier Transform Pairs

TABLE 4.2 BASIC FOURIER TRANSFORM PAIRS 表 4.2 基本的傅立葉轉換對

Signal	Fourier transform	Fourier series coefficients (if periodic)
$\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$	$2\pi \sum_{k=-\infty}^{+\infty} a_k \delta(\omega - k\omega_0)$	a_k
$e^{j\omega_0 t}$	$2\pi\delta(\omega - \omega_0)$	$a_1 = 1$ $a_k = 0$, otherwise
$\cos \omega_0 t$	$\pi[\delta(\omega - \omega_0) + \delta(\omega + \omega_0)]$	$a_1 = a_{-1} = \frac{1}{2}$ $a_k = 0$, otherwise
$\sin \omega_0 t$	$\frac{\pi}{j}[\delta(\omega - \omega_0) - \delta(\omega + \omega_0)]$	$a_1 = -a_{-1} = \frac{1}{2j}$ $a_k = 0$, otherwise
$x(t) = 1$	$2\pi\delta(\omega)$	$a_0 = 1, \quad a_k = 0, \ k \neq 0$ (this is the Fourier series representation for any choice of $T > 0$)
Periodic square wave $x(t) = \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & T_1 < \lvert t\rvert \leq \frac{T}{2} \end{cases}$ and $x(t + T) = x(t)$	$\sum_{k=-\infty}^{+\infty} \frac{2 \sin k\omega_0 T_1}{k} \delta(\omega - k\omega_0)$	$\frac{\omega_0 T_1}{\pi} \operatorname{sinc}\left(\frac{k\omega_0 T_1}{\pi}\right) = \frac{\sin k\omega_0 T_1}{k\pi}$
$\sum_{n=-\infty}^{+\infty} \delta(t - nT)$	$\frac{2\pi}{T} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{T}\right)$	$a_k = \frac{1}{T}$ for all k
$x(t) \begin{cases} 1, & \lvert t\rvert < T_1 \\ 0, & \lvert t\rvert > T_1 \end{cases}$	$\frac{2 \sin \omega T_1}{\omega}$	—
$\frac{\sin Wt}{\pi t}$	$X(j\omega) = \begin{cases} 1, & \lvert\omega\rvert < W \\ 0, & \lvert\omega\rvert > W \end{cases}$	—
$\delta(t)$	1	—
$u(t)$	$\frac{1}{j\omega} + \pi\delta(\omega)$	—
$\delta(t - t_0)$	$e^{-j\omega t_0}$	—
$e^{-at}u(t), \ \mathcal{Re}\{a\} > 0$	$\frac{1}{a + j\omega}$	—
$te^{-at}u(t), \ \mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^2}$	—
$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t)$, $\mathcal{Re}\{a\} > 0$	$\frac{1}{(a + j\omega)^n}$	—

4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

A particularly important and useful class of continuous-time LTI systems is those for which the input and output satisfy a linear constant-coefficient differential equation of the form

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}. \qquad (4.72)$$

連續時間N階線性常係數微分方程一般式

4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

There are two closely related ways in which to determine the frequency response *H(jω)* for an LTI system described by the differential equation (4.72).

The first way is use the fact that $x(t) = e^{j\omega t}$ is the **eigenfunction** of a LTI system, the output must be $y(t) = H(j\omega)e^{j\omega t}$

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

Alternatively, consider an LTI system characterized by eq. (4.72). From the convolution property,

$$Y(j\omega) = H(j\omega)X(j\omega),$$

or equivalently,

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}, \tag{4.73}$$

另一為利用迴旋運算定理可得：

$$Y(j\omega) = H(j\omega)X(j\omega)$$

即：

$$H(j\omega) = Y(j\omega)/X(j\omega)$$

4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

Consider applying the Fourier transform to both sides of eq. (4.72) to obtain

$$F\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = F\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}. \qquad (4.74)$$

直接對微分方程各項求取傅立葉轉換。

4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

From the linearity property, eq. (4.26), this becomes

$$\sum_{k=0}^{N} a_k F\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k F\left\{\frac{d^k x(t)}{dt^k}\right\}, \qquad (4.75)$$

and from the differentiation property, eq. (4.31),

$$\sum_{k=0}^{N} a_k (j\omega)^k Y(j\omega) = \sum_{k=0}^{M} b_k (j\omega)^k X(j\omega),$$

4.7 Systems Characterized by Linear Constant-Coefficient Differential Equations

Or equivalently,

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{M} b_k (j\omega)^k\right].$$

Thus, from eq. (4.73),

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}. \qquad (4.76)$$

可得頻率響應$H(j\omega)$與微分方程各係數的關係式。

Example 4.24 $H(j\omega) = \dfrac{Y(j\omega)}{X(j\omega)} = \dfrac{\sum_{k=0}^{M} b_k (j\omega)^k}{\sum_{k=0}^{N} a_k (j\omega)^k}$.(4.76)

Consider a table LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), \tag{4.77}$$

with $a > 0$. From eq. (4.76), the frequency response is

$$H(j\omega) = \frac{1}{j\omega + a}. \tag{4.78}$$

Comparing this with the result of Example 4.1, we see that eq. (4.78) is the Fourier transform of $e^{-at}u(t)$. The impulse response of the system is then recognized as

$$h(t) = e^{-at}u(t).$$

4.8 Summary

- Derive FT for aperiodic signal from FS
- Convergence of FT
- FT for periodic signals
- Properties of FT: linearity, time-shifting, Conjugate symmetry, differentiation & integration, duality, Parseval's relation, etc.
- Convolution & multiplication properties
- Frequency-Selective Filtering with Variable Center Frequency
- Solving Linear Constant-Coefficient Differential Equations using FT properties