Chapter 3
Fourier series Representation of Periodic Signals

Min sun

3.0 Introduction

- In this chapter, we focus on the representation of continuous-time and discrete-time **periodic** signals referred to as the **Fourier series**. In Chapters 4 and 5, we extend the analysis to the Fourier transform representation of broad classes of **aperiodic, finite energy signals**.

  本章將焦點置於連續時間與離散時間週期訊號的傅立葉級數表示法，第4及5章再將它展至非週期訊號

3.0 Introduction

- These representations provide one of the most powerful and important sets of tools and insights for analyzing, designing, and understanding signals and **LTI systems**, and we devote considerable attention in this and subsequent chapters to exploring the uses of Fourier methods.

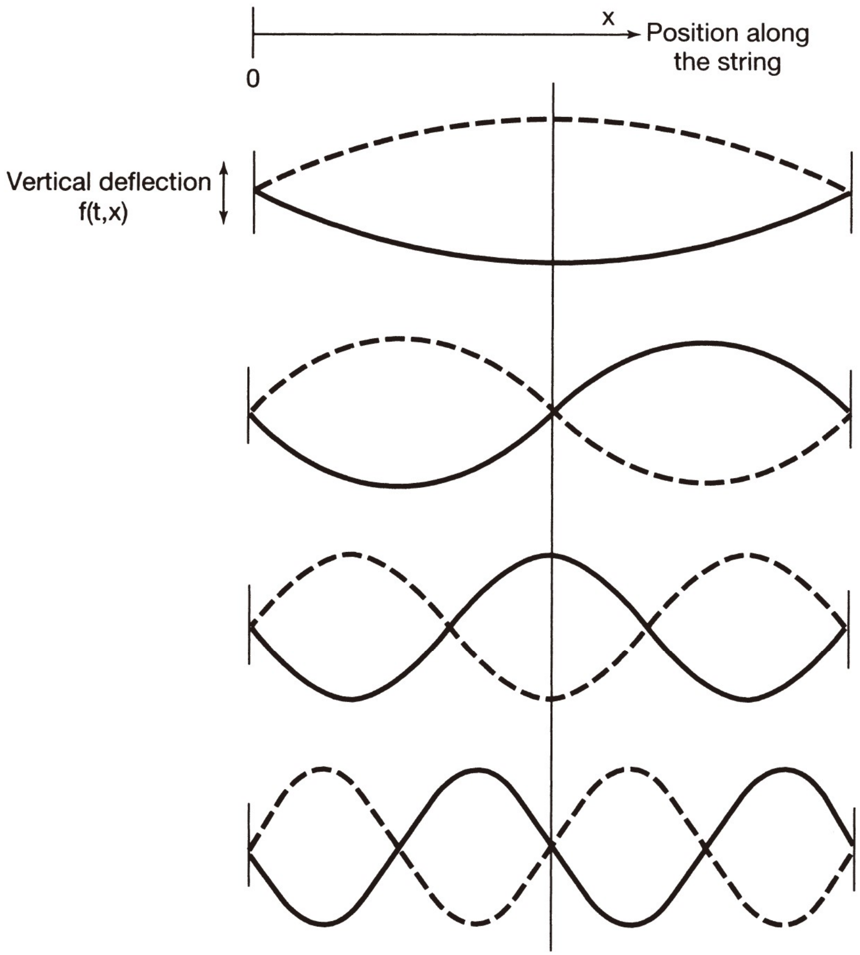
這些表示法將是我們對訊號與LTI系統在分析、設計和理解上極有用而重要的工具。

3.1 A Historical Perspective

- We will see that if the input to an LTI system is expressed as a **linear combination of periodic complex exponentials or sinusoids**, the output can also be expressed in this form, with coefficients that are related in a straightforward way to those of the input.

  若一個LTI系統的輸入可表為數個週期性的複指數或弦波訊號的線性組合，則其(穩態)總輸出亦可利用各輸入相對的輸出透過相同的係數組合。

3.1 A Historical Perspective



1748 L. Euler studied "normal modes" of Vibrating string.
Give any t f(t,x) is harmonically related to sinusoidal function of x*

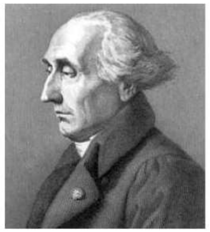
*a set of periodic function with fundamental frequencies that are all multiples of a single positive frequency.

Figure 3.1 Normal modes of a vibrating string. (Solid lines indicate the configuration of each of these modes at some fixed instant of time, *t*.)

3.1 A Historical Perspective

Any univariate function can be rewritten as a weighted sum of sines and cosines of different Frequencies (1807)

Laplace



Lagrange

Legendre

Poisson

...the manner in which the author arrives at these equations is not exempt of difficulties and...his analysis to integrate them still leaves something to be desired on the score of generality and even rigour.

- Laplace

Not translated to English until 1878!

3.1 A Historical Perspective

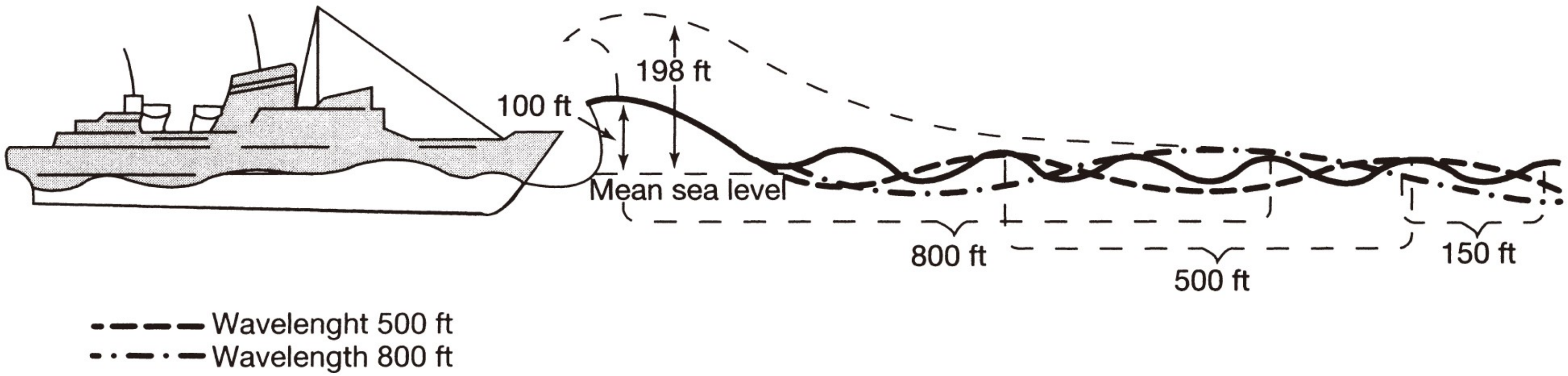


Figure 3.3 Ship encountering the superposition of three wave trains, each with a different spatial period. When these waves reinforce one another, a very large wave can result. In more severe seas, a giant wave indicated by the dotted line could result. Whether such a reinforcement occurs at any location depends upon the relative phases of the components that are superposed. [Adapted from an illustration by P. Mion in "Nightmare Waves Are All Too Real to Deepwater Sailors," by P. Britton, *Smithsonian* 8 (February 1978), pp. 64–65].

3.2 The Response of LTI Systems to Complex Exponentials

- It is advantageous in the study of LTI systems to represent signals as **linear combinations** of **basic** signals that possess the following two properties:

1. The set of basic signals can be used to construct a broad and useful class of signals.
2. The response of an LTI system to each signal should be simple enough in structure to provide us with a convenient representation for the response of the system to any signal constructed as a linear combination of the basic signals.

3.2 The Response of LTI Systems to Complex Exponentials

- The importance of complex exponentials stems from the fact that the response of an LTI system to a complex exponential input is the same complex exponential with only a change in amplitude; that is,

continuous time: $e^{st} \rightarrow H(s)e^{st}$, (3.1)

discrete time: $z^{n} \rightarrow H(z)z^{n}$, (3.2)

where the complex amplitude factor H(s) or H(z) will in general be a function of the complex variable s or z.

3.2 The Response of LTI Systems to Complex Exponentials

- To show that complex exponentials are indeed **eigenfunctions** of LTI systems, let us consider a continuous-time LTI system with impulse response $h(t)$. For an input $x(t)$, we can determine the output through the use of the convolution integral, so that with $x(t) = e^{st}$

一個訊號對系統的輸出正好是此輸入訊號乘以某一常數，則此訊號(函數)為系統的「特徵函數」，且振幅的因數為系統的「特徵值」。

3.2 The Response of LTI Systems to Complex Exponentials

$$x(t) = e^{st}$$

$$y(t) = \int_{-\infty}^{+\infty} h(\tau)x(t-\tau)d\tau$$

(3.3)

$$= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau.$$

(3.4)

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau.$$

Expressing $e^{s(t-\tau)}$ as $e^{st}e^{-s\tau}$, and noting that e^{st} can be moved outside the integral

3.2 The Response of LTI Systems to Complex Exponentials

$$y(t) = e^{st}\int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau.$$

The response to e^{st} is of the form

$$y(t) = H(s)e^{st} \qquad (3.5)$$

Where H(s) is a complex constant whose value depends on s ant which is related to the system impulse response by

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \qquad (3.6)$$

3.2 The Response of LTI Systems to Complex Exponentials

Hence, we have shown that complex exponentials are eigenfunctions of LTI systems. The constant *H*(s) for a specific value of *s* is then the **eigenvalue** associated With the eigenfunction e^{st} .

3.2 The Response of LTI Systems to Complex Exponentials

Suppose that ant LTI system with impulse response $h[n]$ has as its input the sequence $x[n] = z^n$ (3.7)

Where z is a complex number. Then the output of the system can be determined from the convolution sum as

$$y[n] = \sum_{k=-\infty}^{+\infty} h[k]x[n-k]$$

$$= \sum_{k=-\infty}^{+\infty} h[k]z^{n-k} = z^n \sum_{k=-\infty}^{+\infty} h[k]z^{-k}. \qquad (3.8)$$

3.2 The Response of LTI Systems to Complex Exponentials

$$y[n] = z^n \sum_{k=-\infty}^{+\infty} h[k] z^{-k}.$$

The output is the same complex exponential multiplied by a constant that depends on the value of z. That is,

$$y[n] = H(z) z^n, \qquad (3.9)$$

where

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k}. \qquad (3.10)$$

3.2 The Response of LTI Systems to Complex Exponentials

Consequently, as in the continuous-time case, z^n complex exponentials are **eigenfunctions** of discrete-time LTI systems. The constant $H(z)$ for a specified value of z is the **eigenvalue** associated with the eigenfunction z^n.

3.2 The Response of LTI Systems to Complex Exponentials

Let *x(t)* correspond to a linear combination of three complex exponentials; that is,

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}. \quad (3.11)$$

From the eigenfunction property, the response to each separately is

$$a_1 e^{s_1 t} \rightarrow a_1 H(s_1) e^{s_1 t},$$

$$a_2 e^{s_2 t} \rightarrow a_2 H(s_2) e^{s_2 t},$$

$$a_3 e^{s_3 t} \rightarrow a_3 H(s_3) e^{s_3 t},$$

3.2 The Response of LTI Systems to Complex Exponentials

and from the superposition property the reponse to the sum is the sum of the responses, so that

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}.$$

(3.12)

3.2 The Response of LTI Systems to Complex Exponentials

If the input to a continuous-time LTI system is represented as a linear combination of complex exponentials, that is, if

$$x(t) = \sum_k a_k e^{s_k t}, \tag{3.13}$$

then the output will be

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}. \tag{3.14}$$

3.2 The Response of LTI Systems to Complex Exponentials

If

$$x[n] = \sum_k a_k z_k^n, \tag{3.15}$$

then the output will be

$$y[n] = \sum_k a_k H(z_k) z_k^n. \tag{3.16}$$

Example 3.1

$$y(t) = H(s)e^{st} \quad (3.5)$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \quad (3.6)$$

As an illustration of Eqs. (3.5) and (3.6), consider an LTI system for which the input *x(t)* and output *y(t)* are related by a time shift of 3,i.e.,

(3.17)

$$y(t) = x(t-3).$$

If the input to this system is the complex exponential signal $x(t) = e^{j2t}$, then, from eq.(3.17),

(3.18)

$$y(t) = e^{j2(t-3)} = e^{-j6}e^{j2t}.$$

Example 3.1

$$y(t) = H(s)e^{st} \quad (3.5) \qquad y(t) = e^{-j6}e^{j2t} \quad (3.18)$$

$$H(s) = \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau}d\tau \quad (3.6) \qquad y(t) = x(t-3) \quad (3.17)$$

Equation (3.18) is in the form of eq. (3.5), as we would expect, since e^{j2t} is an eigenfunction. The associated eigenvalue is $H(j2) = e^{-j6}$. It is straightforward to confirm eq. (3.6) for this example. Specifically, from eq. (3.17), the impulse response of the system is $h(t) = \delta(t-3)$. Substituting into eq. (3.6), we obtain

$$H(s) = \int_{-\infty}^{+\infty} \delta(\tau - 3)e^{-s\tau}d\tau = e^{-3s},$$

so that $H(j2) = e^{-j6}$.

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t} \quad (3.11)$$

Example 3.1

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t} \quad (3.12)$$

$$y(t) = x(t-3) \quad (3.17)$$

As a second example, in this case illustrating eqs. (3.11) and (3.12), consider the input signal $x(t) = \cos(4t) + \cos(7t)$. From eq. (3.17), *y(t)* will of course be

$$y(t) = \cos(4(t-3)) + \cos(7(t-3)). \quad (3.19)$$

To see that this will also result from eq. (3.12), we first expand *x(t)* using Euler's relation:

$$x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-4t} + \frac{1}{2} e^{j7t} + \frac{1}{2} e^{-j7t}. \quad (3.20)$$

$$H(s) = e^{-3s}$$

Example 3.1

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t} \quad (3.11)$$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t} \quad (3.12)$$

Given $x(t) = \frac{1}{2} e^{j4t} + \frac{1}{2} e^{-4t} + \frac{1}{2} e^{j7t} + \frac{1}{2} e^{-j7t}.$

From eqs. (3.11) and (3.12),

$$y(t) = \frac{1}{2} e^{-j12} e^{j4t} + \frac{1}{2} e^{j12} e^{-j4t} + \frac{1}{2} e^{-j21} e^{j7t} + \frac{1}{2} e^{j21} e^{-j7t},$$

or

$$y(t) = \frac{1}{2} e^{j4(t-3)} + \frac{1}{2} e^{-j4(t-3)} + \frac{1}{2} e^{j7(t-3)} + \frac{1}{2} e^{-j7(t-3)}$$

$$= \cos(4(t-3)) + \cos(7(t-3)). \qquad (3.19)$$

Example 3.1

$$y(t) = \cos(4(t-3)) + \cos(7(t-3))(3.19)$$

$$x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}(3.11)$$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}(3.12)$$

In this case we can determine *y(t)* in eq. (3.19) by inspection rather than by employing eqs. (3.11) and (3.12)

However, eqs. (3.11) and (3.12) not only allows us to calculate the responses of more **complex** LTI systems, but also provides the basis for the frequency domain representation and analysis of LTI systems.

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a signal is periodic if, for some positive value of T,

(3.21) $$x(t) = x(t+T)$$ for all t.

The fundamental period of *x(t)* is the **minimum positive, nonzero** value of T for which eq. (3.21) is satisfied, and the value $\omega_0 = 2\pi / T$ is referred to as the fundamental frequency (基本頻率).

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

Two basic periodic signals, the sinusoidal signal

弦波訊號 $$x(t) = \cos \omega_0 t \qquad (3.22)$$

and the periodic complex exponential

週期複指數 $$x(t) = e^{j\omega_0 t}. \qquad (3.23)$$

Both of these signals are periodic with fundamental frequency ω_0 and fundamental period $T = 2\pi / \omega_0$. Associated with the signal in eq. (3.23) is the set of **harmonically related** complex exponentials*

$$\phi_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi / T)t}, k = 0, \pm 1, \pm 2, \ldots\ldots$$

*a set of periodic exponentials with fundamental frequencies that are all multiples of ω_0

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

- **Fourier series representation**:

A linear combination of harmonically related complex exponentials of the form

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk(2\pi/T)t} \qquad (3.25)$$

is **also periodic with period *T***.

In eq. (3.25), the term for k=0 is a constant. The terms for k= +1 and k= -1 both have fundamental frequency equal to ω_0 and are collectively referred to as the *fundamental components* for the ***first harmonic components***.

More generally, the components for k= +N and k= -N are referred to as the **N^{th} harmonic components**.

Example 3.2

Consider a periodic signal *x(t)*, with fundamental frequency 2π, that is expressed in the form of eq. (3.25) as

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}, \qquad (3.26)$$

where

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

Example 3.2

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t} \quad (3.26)$$

Rewriting eq. (3.26) and collecting each of the harmonic components which have the same fundamental frequency, we obtain

$$x(t) = 1 + \frac{1}{4}(e^{j2\pi t} + e^{-j2\pi t}) + \frac{1}{2}(e^{j4\pi t} + e^{-j4\pi t}) + \frac{1}{3}(e^{j6\pi t} + e^{-j6\pi t}). \quad (3.27)$$

Equivalently, using Euler's relation, we can write x(t) in the form

$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t. \quad (3.28)$$

* alternative form for Fourier Series of real periodic signal

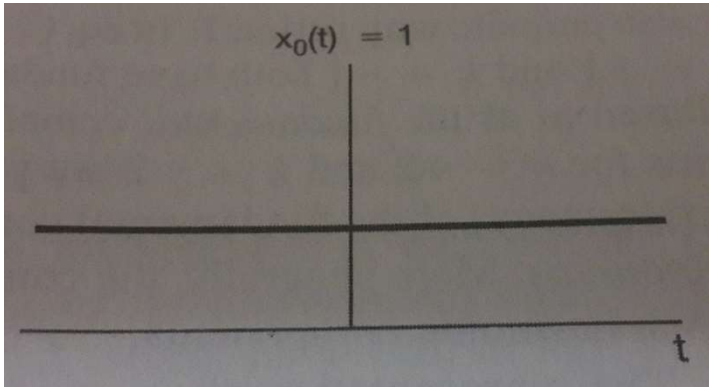
Example 3.2

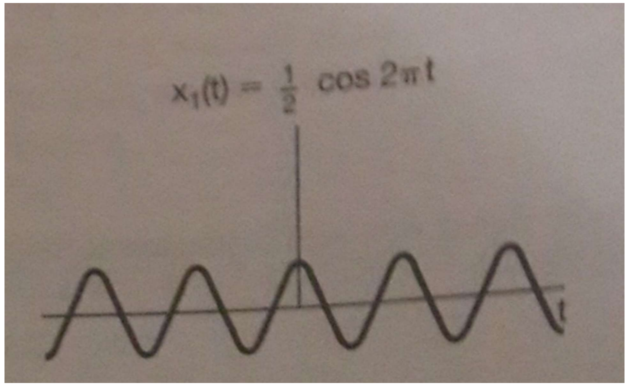
$$x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t.$$

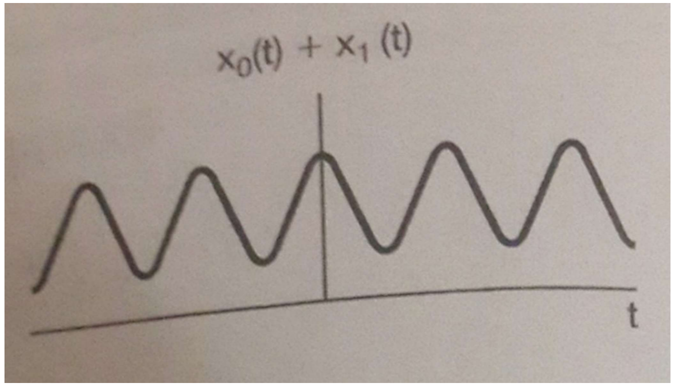
In figure 3.4, we illustrate graphically how the signal *x(t)* is built up from its harmonic components.

Individual signal

Combined signal







Example 3.2

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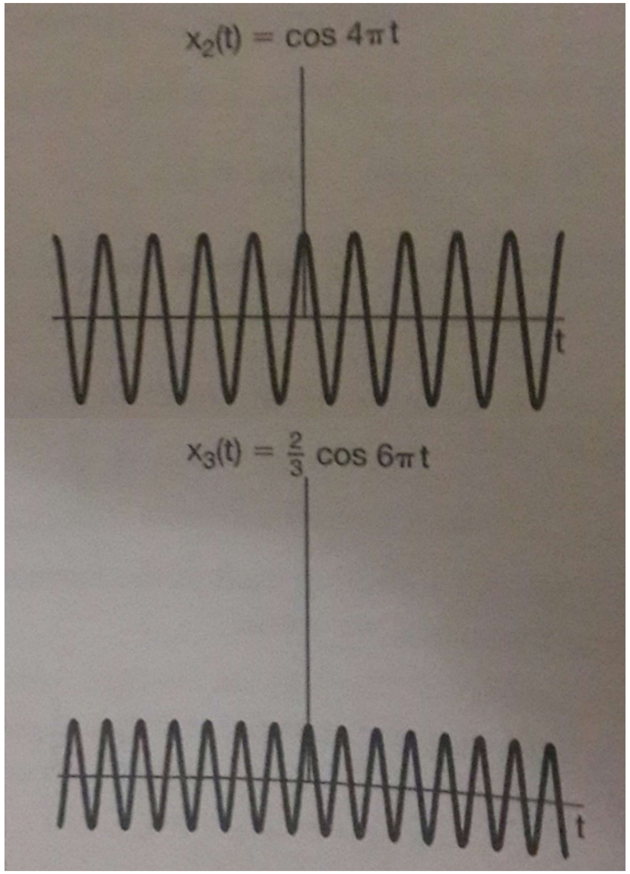
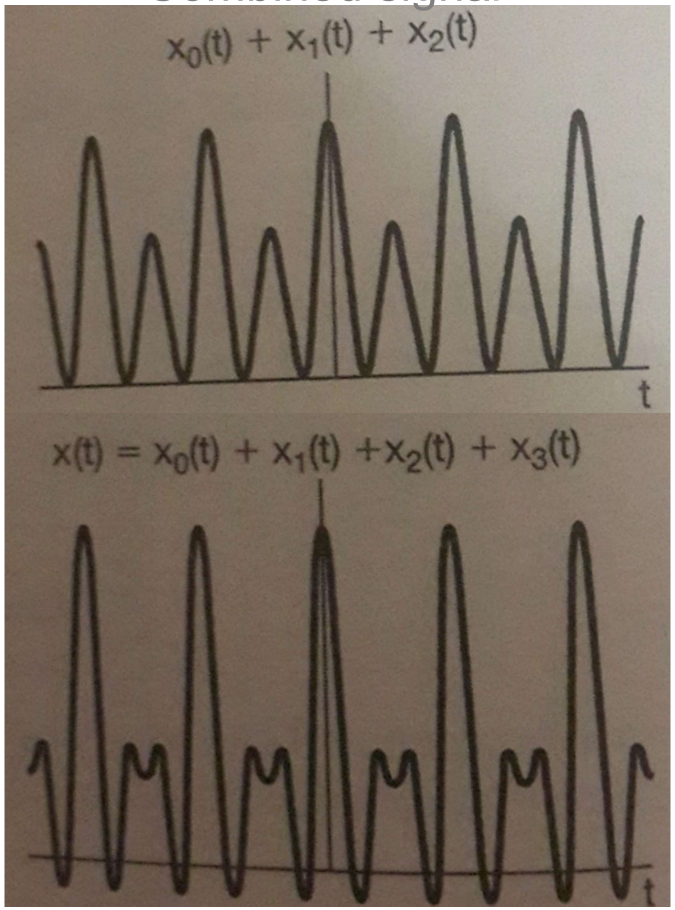
Individual signal



Combined signal



3.3.1 Linear combinations of Harmonically Related Complex Exponentials

- Alternative form for Fourier Series of **real** periodic signal

  Suppose that x(t) is real and can be represented in the form of eq. (3.25). Then, since $x^*(t) = x(t)$, we obtain

$$x(t) = x^*(t) = \sum_{k=-\infty}^{+\infty} a_k^* e^{-jk\omega_0 t}. \qquad x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (3.25)$$

  Replacing k by – k in the summation, we have

$$x(t) = \sum_{k=-\infty}^{+\infty} a_{-k}^* e^{jk\omega_0 t},$$

  by comparison with eq. (3.25), requires that $a_k = a_{-k}^*$, or equivalently, that

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (3.25)$$

To derive the alternative forms of the Fourier series, we first rearrange the summation in eq. (3.25) as

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t} \right]$$

Substituting a_k^* for a_{-k} from eq. (3.29), we obtain

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_k^* e^{-jk\omega_0 t} \right].$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e \left\{ a_k e^{jk\omega_0 t} \right\}.$$

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

Since the two terms inside the summation are complex conjugates of each other, this can be expressed as

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\left\{a_k e^{jk\omega_0 t}\right\}. \tag{3.30}$$

If a_k is expressed in **polar** form as $a_k = A_k e^{j\theta_k}$,

then eq. (3.30) becomes

That is, $$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\left\{A_k e^{j(k\omega_0 t + \theta_k)}\right\}$$

$$x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k). \tag{3.31}$$

3.3.1 Linear combinations of Harmonically Related Complex Exponentials

Another form is obtained by writing a_k in **rectangular** form as

$$a_k = B_k + jC_k$$

where B_k and C_k are both real. With this expression for a_k , eq. (3.30) takes the form

$$x(t) = a_0 + 2\sum_{k=1}^{\infty}\left[B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right]. \quad (3.32)$$

$$x(t) = a_0 + \sum_{k=1}^{\infty} 2\Re e\left\{a_k e^{jk\omega_0 t}\right\} = a_0 + \sum_{k=1}^{\infty} 2(\Re e\left\{B_k e^{jk\omega_0 t}\right\} + \Re e\left\{jC_k e^{jk\omega_0 t}\right\})$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

- Given x(t) and ω_0 , how to determine a_k?

Multiplying both sides of eq. (3.25) by $e^{-jn\omega_0 t}$, we obtain

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t}.$$

Integrating both sides from 0 to $T = 2\pi / \omega_0$, we have

$$\int_0^T x(t)e^{-jn\omega_0 t} dt = \int_0^T \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} e^{-jn\omega_0 t} dt.$$

T is the fundamental period of *x(t)*,

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

Interchanging the order of integration and summation yields

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \sum_{k=-\infty}^{+\infty} a_k\left[\int_0^T e^{j(k-n)\omega_0 t}\,dt\right]. \quad (3.34)$$

Rewriting this integral using Euler's formula, we obtain

$$\int_0^T e^{j(k-n)\omega_0 t}dt = \int_0^T \cos(k-n)\omega_0 t dt + j\int_0^T \sin(k-n)\omega_0 t dt. \quad (3.35)$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

For k ≠ n, $\cos(k-n)\omega_0 t$ and $\sin(k-n)\omega_0 t$ are periodic sinusoids with fundamental period (T/ | k-n |). Therefore, eq. (3.35) equals zero. For k = n, the integrand on the left-hand side of eq. (3.35) equals 1, and thus, the integral equals *T*. In sum, we then have

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, k=n \\ 0, k \neq n \end{cases}$$

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \int_0^T \cos(k-n)\omega_0 t\, dt + j\int_0^T \sin(k-n)\omega_0 t\, dt. \quad (3.35)$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

$$\int_0^T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, k=n \\ 0, k\neq n \end{cases}$$

$$\int_0^T x(t)e^{-jn\omega_0 t} dt = \sum_{k=-\infty}^{+\infty} a_k \left[\int_0^T e^{j(k-n)\omega_0 t} \, dt \right] = \sum_{k=-\infty}^{+\infty} a_k T\delta[k-n]$$

the right-hand side of eq. (3.34) reduces to Ta_n

Therefore,

$$a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t} dt, \qquad (3.36)$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

if we denote integration over any interval of length T by $\int_T$, we have

$$\int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k=n \\ 0, & k \neq n \end{cases},$$

and consequently,

$$a_n = \frac{1}{T} \int_T x(t) e^{-jn\omega_0 t} dt. \qquad (3.37)$$

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

defines the Fourier series of a periodic continuous-time signal:

$$x(t)=\sum_{k=-\infty}^{+\infty}a_k e^{jk\omega_0 t}=\sum_{k=-\infty}^{+\infty}a_k e^{jk(2\pi/T)t}, \quad (3.38)$$

$$a_k=\frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt=\frac{1}{T}\int_T x(t)e^{-jk(2\pi/T)t}dt. \quad (3.39)$$

$\{a_k\}$ Fourier coefficients

3.3.2 Determination of the Fourier Series Representation of a Continuous-time Periodic Signal

The set of coefficients $\{a_k\}$ are often called the **Fourier series coefficients** or the **spectral coefficients** of *x(t).*

The coefficient a_0 is the dc or constant component of *x(t)* and is given by eq. (3.39) with k = 0. That is,

$$a_0 = \frac{1}{T}\int_T x(t)dt, \qquad (3.40)$$

$$a_n = \frac{1}{T}\int_T x(t)e^{-jn\omega_0 t}dt.$$

Example 3.4

Let

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right),$$

Which has fundamental frequency ω_0. As Example 3.3, we can again expand x(t) directly in terms of complex exponentials, so that

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left[e^{j(2\omega_0 t + \pi/4)} + e^{-j(2\omega_0 t + \pi/4)}\right].$$

Collecting terms, we obtain

$$x(t) = 1 + \left(1 + \frac{1}{2j}\right)e^{j\omega_0 t} + \left(1 - \frac{1}{2j}\right)e^{-j\omega_0 t} + \left(\frac{1}{2}e^{j(\pi/4)}\right)e^{j2\omega_0 t} + \left(\frac{1}{2}e^{-j(\pi/4)}\right)e^{-j2\omega_0 t}.$$

Example 3.4

Thus, the Fourier series coefficients for this example are

$$a_0 = 1,$$

$$a_1 = \left(1 + \frac{1}{2j}\right) = 1 - \frac{1}{2}j,$$

$$a_{-1} = \left(1 - \frac{1}{2j}\right) = 1 + \frac{1}{2}j,$$

$$a_2 = \frac{1}{2}e^{j(\pi/4)} = \frac{\sqrt{2}}{4}(1 + j),$$

$$a_{-2} = \frac{1}{2}e^{-j(\pi/4)} = \frac{\sqrt{2}}{4}(1 - j),$$

$$a_k = 0, \qquad |k| \rangle 2.$$

Example 3.4

In Figure 3.5, we show a bar graph of the magnitude and phase of a_k .

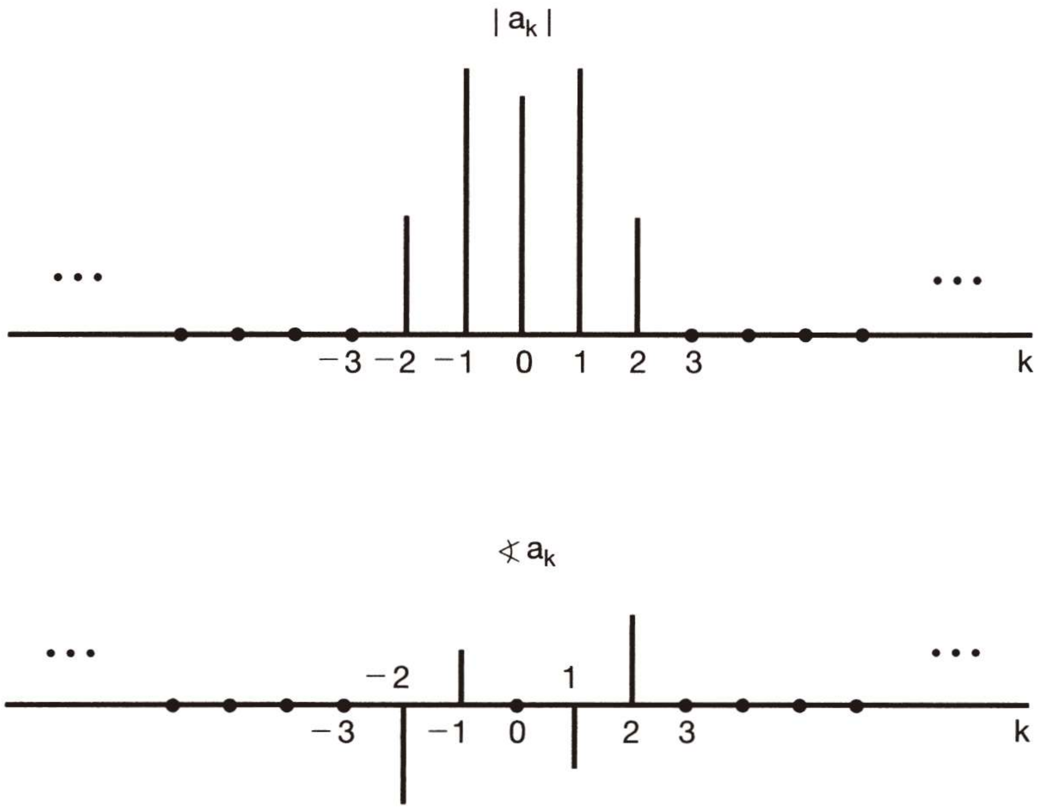


Figure 3.5 Plots of the magnitude and phase of the Fourier coefficients of the signal considered in Example 3.4.

3.4 Convergence of the Fourier Series

- Can any periodic signal x(t) be represented by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (3.38)$$

a linear combination of a **infinite** number of harmonically related complex exponentials?

Let us define $x_N(t)$ as a **finite** series of the form

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}. \quad (3.47)$$

3.4 Convergence of the Fourier Series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}.$$

Let $e_N(t)$ denote the approximation error; that is,

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \quad (3.48)$$

The quantitative measure of the approximation error is defined by the **energy in the error over one period**:

$$E_N = \int_T |e_N(t)|^2 dt. \quad (3.49)$$

3.4 Convergence of the Fourier Series

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \quad (3.47) \qquad E_N = \int_T |e_N(t)|^2 dt \quad (3.49)$$

As shown in Problem 3.66, the particular choice for the coefficients in eq. (3.47) that minimize the energy in the error is

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt. \qquad (3.50)$$

We consider x(t) has a valid Fourier series representation if E_N is zero in the limit when $N \longrightarrow \infty$

When will this be true?

3.4 Convergence of the Fourier Series

One class of periodic signals that are representable through the Fourier series is those signals which have **finite energy over a single period**, i.e., signals for which

$$\int_T |x(t)|^2 dt < \infty. \tag{3.51}$$

$x_N(t)$ be the approximation to *x(t)* obtained by using these coefficients for $|k| \le N$:

$$x_N(t) = \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \tag{3.52}$$

3.4 Convergence of the Fourier Series

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt.$$

$N \to \infty$. That is, if we define

$$e(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}. \qquad (3.53)$$

then

$$\int_T |e(t)|^2 dt = 0. \qquad (3.54)$$

Note signal x(t) and its Fourier series representation are equal at all t

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad for \quad all \quad t$$

3.4 Convergence of the Fourier Series

$$\int_T |e_{N+1}(t)|^2 dt \le \int_T |e_N(t)|^2 dt \quad for \quad all \quad N$$

$$e_N(t) = x(t) - \sum_{k=-N}^{+N} a_k e^{jk\omega_0 t}; e_{N+1}(t) = x(t) - \sum_{k=-N-1}^{+N+1} a_k e^{jk\omega_0 t};$$

$$e_{N+1}(t) = e_N(t) - a_{-N-1} e^{-j(N+1)\omega_0 t} - a_{N+1} e^{j(N+1)\omega_0 t}$$

$$\int_T e^{j(k-n)\omega_0 t} dt = \begin{cases} T, & k=n \\ 0, & k \ne n \end{cases}$$

$$\int_T |e_{N+1}(t)|^2 dt = \int_T |e_N(t)|^2 dt - T(|a_{-N-1}|^2 + |a_{N+1}|^2)$$

$$\int_T |e_{N+1}(t)|^2 dt \le \int_T |e_N(t)|^2 dt \le \int_T |e_0(t)|^2 dt \le \int_T |x(t)|^2 dt < \infty$$

迪利斯雷(Dirichlet)條件：

x(t) equals its Fourier Series representation, except at isolated values of t for which x(t) is discontinuous.

Condition 1. Over an period, *x(t)* must be *absolutely integrable*; that is,

$$\int_T |x(t)| dt \langle \infty. \qquad (3.56)$$

條件1:在任何時間區間上，*x(t)*必須為絕對可積分。

迪利斯雷(Dirichlet)條件：

this guarantees that each coefficient will be finite, since

$$|a_k| \le \frac{1}{T}\int_T \left|x(t)e^{-jk\omega_0 t}\right| dt = \frac{1}{T}\int_T |x(t)|\,dt.$$

So if

$$\int_T |x(t)|\,dt \langle \infty,$$

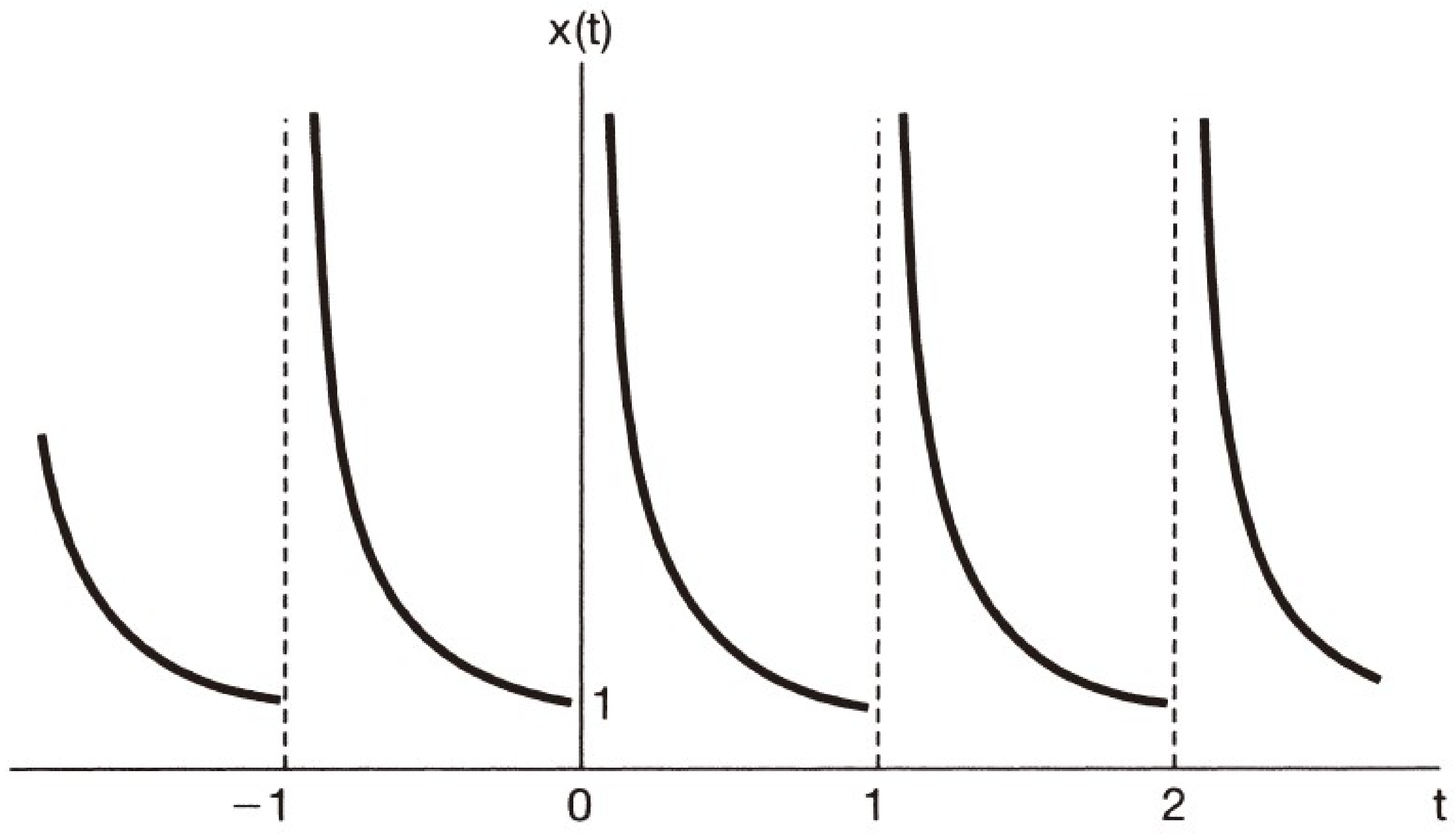
then

$$|a_k| \langle \infty.$$

A periodic signal that violates the first Dirichlet condition is

$$x(t) = \frac{1}{t}, \qquad 0 < t \le 1;$$

3.4 Convergence of the Fourier Series



(a)

迪利斯雷(Dirichlet)條件：

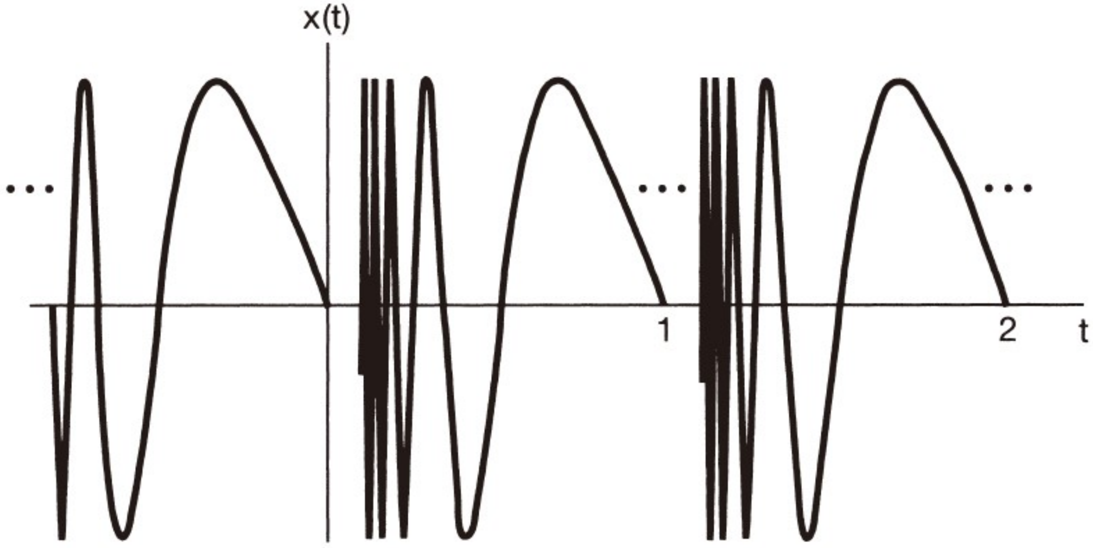
Condition 2. In any finite interval of time, $x(t)$ is of bounded variation; that is, there are no more than a **finite number of maxima and minima** during any single period of the signal.

迪利斯雷(Dirichlet)條件：

A periodic signal that violates the second Dirichlet condition is

$$x(t) = \sin\left(\frac{2\pi}{t}\right), \qquad 0 < t \le 1, \tag{3.57}$$

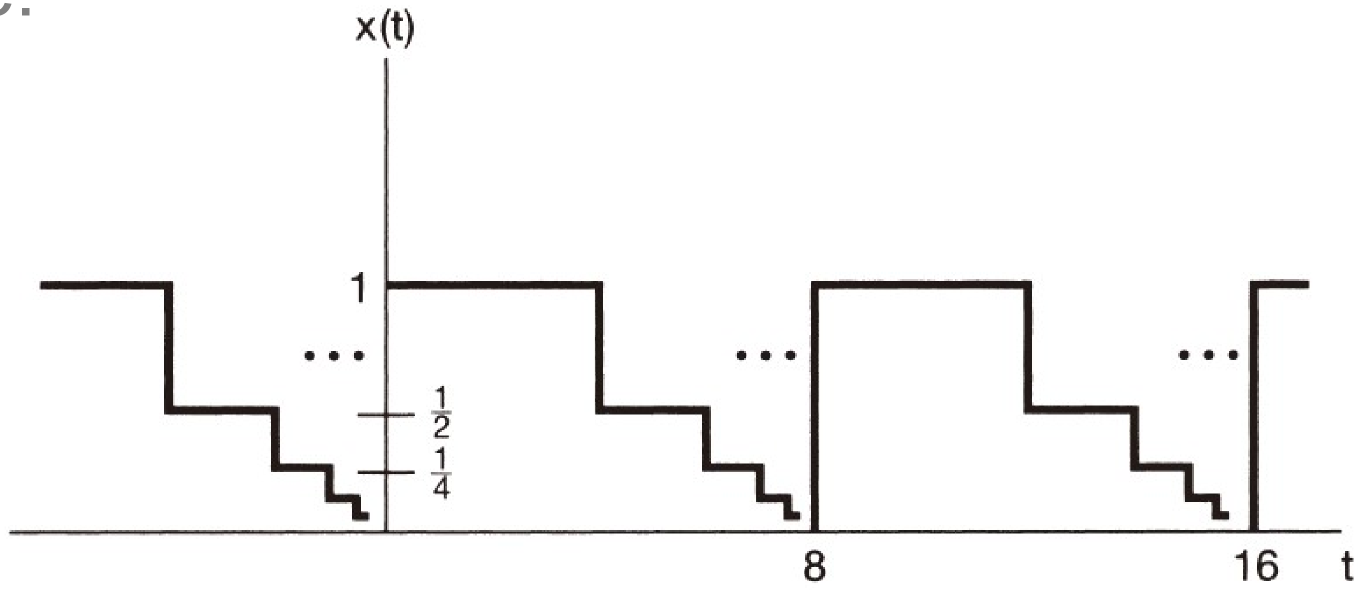
as illustrated in Figure 3.8(b). For this function, which is periodic with *T*=1,



$$\int_0^1 |x(t)| dt < 1.$$

迪利斯雷(Dirichlet)條件：

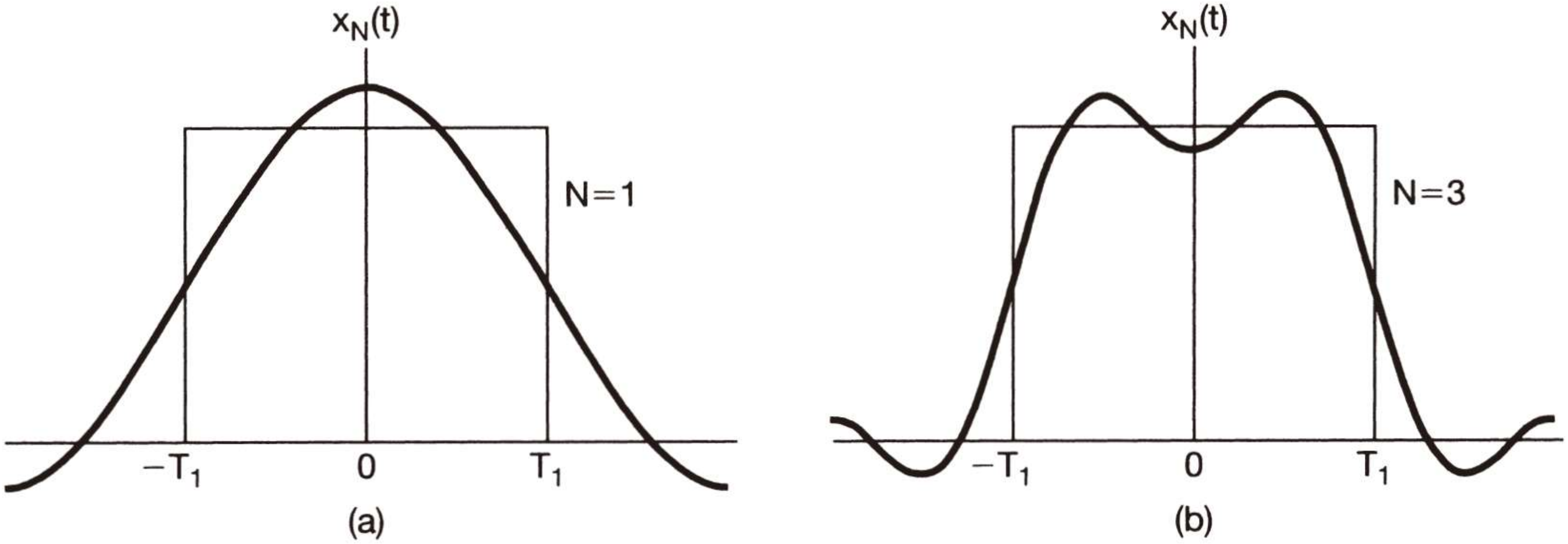
Condition 3. In any finite interval of time, there are only a finite number of discontinuities. Furthermore, each of these discontinuities is finite.



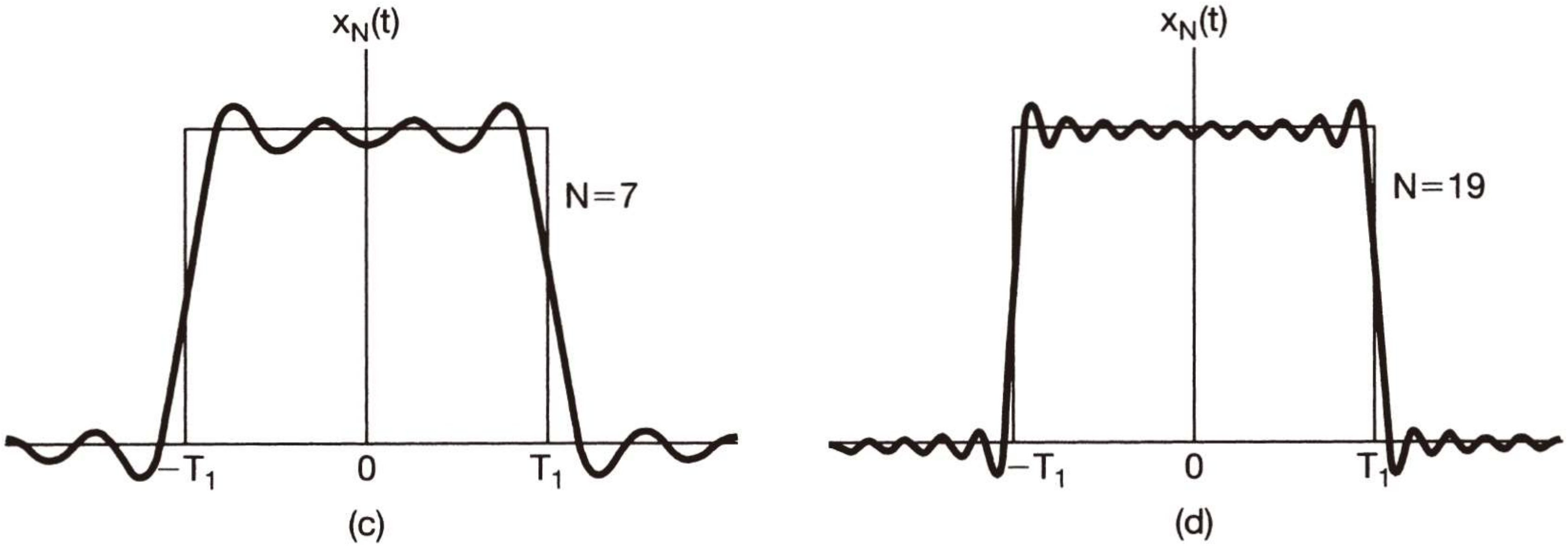
(c)

3.4 Convergence of the Fourier Series

A square wave satisfied all 3 conditions.



3.4 Convergence of the Fourier Series



3.4 Convergence of the Fourier Series

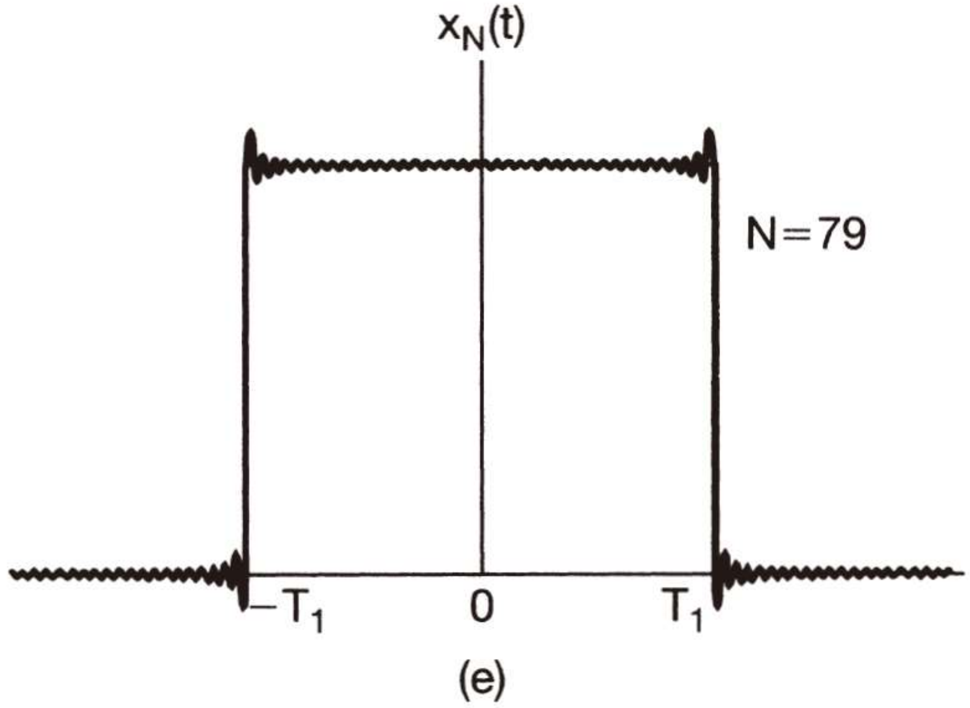


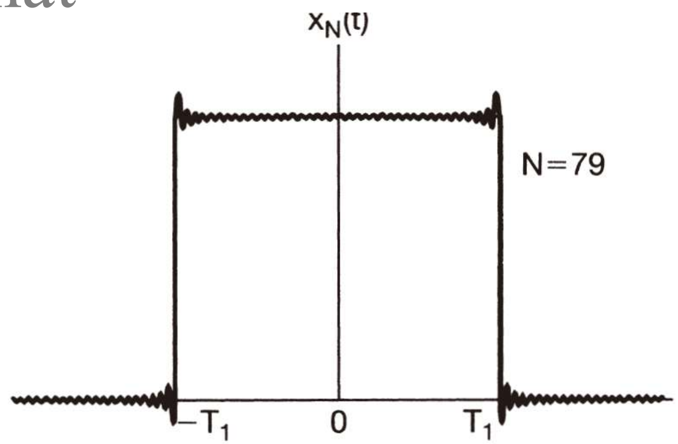
Figure 3.9 Convergence of the Fourier series representation of a square wave: an illustration of the Gibbs phenomenon. Here, we have depicted the finite series approximation $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$ for several values of N.

3.4 Convergence of the Fourier Series

When N->infinite, $x_N(t)$ at discontinuities should be the average value of the discontinuities

We see from the figure that this is in fact the case, since for any N, $x_N(t)$ has exactly that value at the discontinuities. Furthermore, for any other value of t, say, $t = t_1$, we are guaranteed that

$$\lim_{N \to \infty} x_N(t_1) = x(t_1).$$



(e)

3.5 Properties of Continuous-time Fourier series

If the Fourier series coefficients of *x(t)* are denoted by a_k , we will use the notation

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k$$

to signify the pairing of a periodic signal with its Fourier series coefficients.

訊號與其傅立葉係數相互對應的記號。

3.5.1 Linearity

Let *x(t)* and *y(t)* denote two periodic signals **with period T** and which have Fourier Series coefficients denoted by a_k and b_k , respectively. That is,

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k ,$$

$$y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} b_k .$$

3.5.1 Linearity

the Fourier series coefficients c_k of the linear combination of *x(t)* and *y(t)*, *z(t) =Ax(t) + By(t)*

$$z(t) = Ax(t) + By(t) \overset{\Im S}{\longleftrightarrow} c_k = Aa_k + Bb_k.$$

The proof of this follows directly from the application of eq. (3.39). We also note that the linearity property is easily extended to a linear combination of an **arbitrary number of signals with period *T***.

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt \quad (3.39)$$

3.5.2 Time Shifting

The fourier series coefficients b_k of the resulting signal $y(t) = x(t - t_0)$ may be expressed as

$$b_k = \frac{1}{T}\int_T x(t - t_0) e^{-jk\omega_0 t} dt.$$

Letting $\tau = t - t_0$ in the integral, and noting that the new variable τ will also range over an interval of duration *T*, we obtain

$$\frac{1}{T}\int_T x(\tau) e^{-jk\omega_0(\tau + t_0)} d\tau = e^{-jk\omega_0 t_0} \frac{1}{T}\int_T x(\tau) e^{-jk\omega_0 \tau} d\tau \qquad (3.60)$$

$$= e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k,$$

3.5.2 Time Shifting

where a_k is the k^{th} Fourier series coefficient of *x(t).* That is, if

$$x(t) \overset{\Im S}{\longleftrightarrow} a_k,$$

then

$$x(t-t_0) \overset{\Im S}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

Frequency shifting

$$e^{jM\omega_0 t} x(t) \overset{\Im S}{\longleftrightarrow} a_{k-M},$$

3.5.3 Time Reversal

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (3.38)$$

To determine the Fourier series coefficients of *y(t) = x(-t)*, let us consider the effect of time reversal on the synthesis equation (3.38):

$$x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{-jk2\pi t/T}. \quad (3.61)$$

Making the substitution k = -m, we obtain

$$y(t) = x(-t) = \sum_{m=-\infty}^{\infty} a_{-m} e^{jm2\pi t/T}. \quad (3.62)$$

3.5.3 Time Reversal

where the Fourier series coefficients are

$$b_k = a_{-k}. \qquad (3.63)$$

That is, if

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

then

$$x(-t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_{-k}.$$

時間倒轉性質

3.5.4 Time Scaling

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t} \quad (3.38)$$

if *x(t)* has the Fourier series representation in eq. (3.38), then

$$x(\alpha t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk(\alpha\omega_0)t}$$

* the Fourier coefficients are the same, but the fundamental frequency becomes $\alpha\omega_0$

3.5.5 Multiplication

Suppose that *x(t)* and *y(t)* are both periodic with period *T* and that

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

$$y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} b_k.$$

Since x(t)y(t) is also periodic with period T, we can expand it in a Fourier series with Fourier series coefficients h_k expressed in terms of those for *x(t)* and *y(t)*. The result is

$$x(t)y(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}. \qquad (3.64)$$

Periodic convolution

$$\int_T x(\tau)y(t-\tau)d\tau \stackrel{\mathfrak{F}S}{\longleftrightarrow} Ta_k b_k$$

3.5.6 Conjugation and Conjugate Symmetry

Taking the complex conjugate of a periodic signal *x(t)* has the effect of complex conjugation and time reversal on the corresponding Fourier series coefficients. That is, if

$$x(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a_k,$$

then

$$x^*(t) \overset{\mathfrak{F}S}{\longleftrightarrow} a^*_{-k}. \qquad (3.65)$$

$$x^*(t) = \left(\sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}\right)^* = \sum_{k=-\infty}^{+\infty} a^*_k e^{-jk\omega_0 t} = \sum_{m=-\infty}^{+\infty} a^*_{-m} e^{jm\omega_0 t}$$

3.5.6 Conjugation and Conjugate Symmetry

see from eq. (3.65) that the Fourier series coefficients of a **real** signal x(t) will be **conjugate symmetric**, i.e.,

$$a_{-k} = a_k^*, \qquad (3.66)$$

For example, from eq. (3.66), we see that if *x(t)* is real, then a_0 is real and

$$|a_k| = |a_{-k}|.$$

3.5.6 Conjugation and Conjugate Symmetry

If x(t) is real and even

$$x(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_k$$

Real $$x^*(t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_{-k}^* \qquad a_{-k} = a_k^*,$$

Even $$x(-t) \stackrel{\mathfrak{F}S}{\longleftrightarrow} a_{-k} \qquad a_k = a_{-k},$$

Even

$$a_k = a_k^*,$$

Real

3.5.7 Parseval's Relation for Continuous-Time Periodic Signals

As shown n Problem 3.46, Parseval's relation for continuous-time periodic signals is

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_K|^2, \qquad (3.67)$$

Average power

Also,

$$\frac{1}{T}\int_T \left|a_k e^{jk\omega_0 t}\right|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2,$$

is the average power in the k^{th} harmonic component of x(t)

(3.68)

TABLE 3.1 PROPERTIES OF CONTINUOUS-TIME FOURIER SERIES 表 3.1 連續時間傅立葉級數的性質

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} x(t) = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

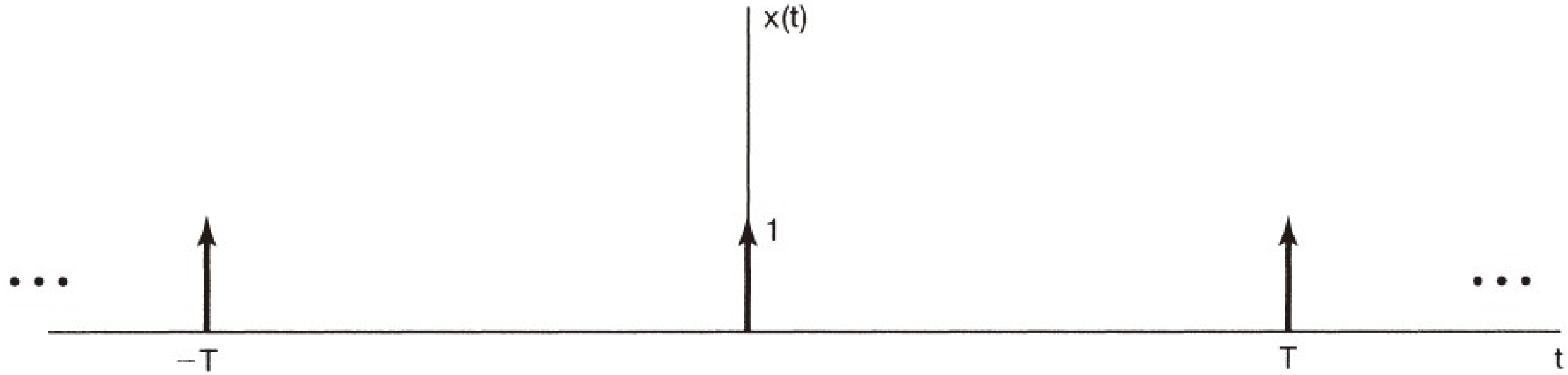
Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Example 3.8

Let us examine some properties of the Fourier series representation of a periodic train of impulses, or *impulse train*. It will play an important role when we discuss the topic of sampling in Chapter 7. The impulse train with period T may be expressed as

$$x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT); \qquad (3.75)$$



Example 3.8

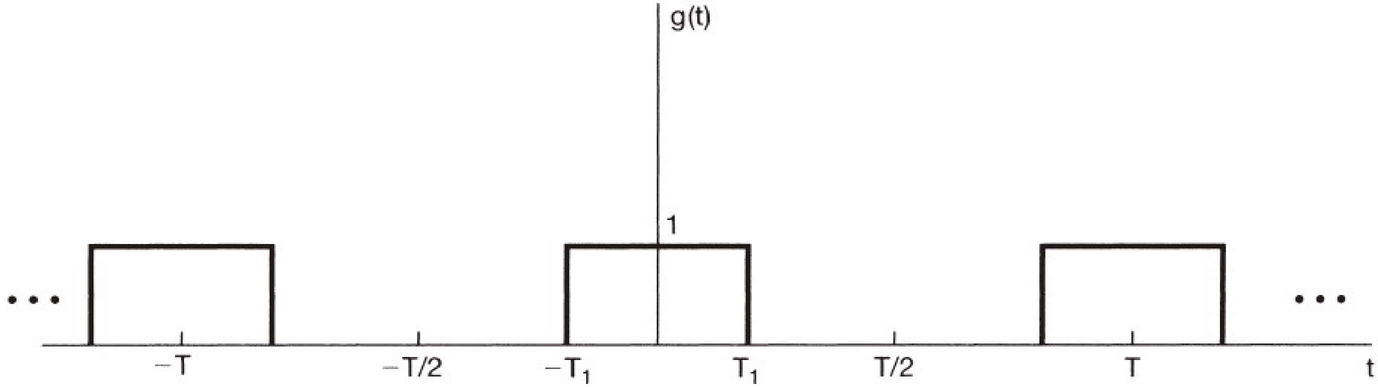
To determine the Fourier series coefficients a_k , we use eq. (3.39) and select the interval of integration to be $-T/2 \le t \le T/2$, avoiding the placement of impulses at the integration limits. Within this interval, x(t) is the same as δ(t), and it follows that

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk2\pi t/T} dt = \frac{1}{T}. \quad (3.76)$$

Constant, real, even

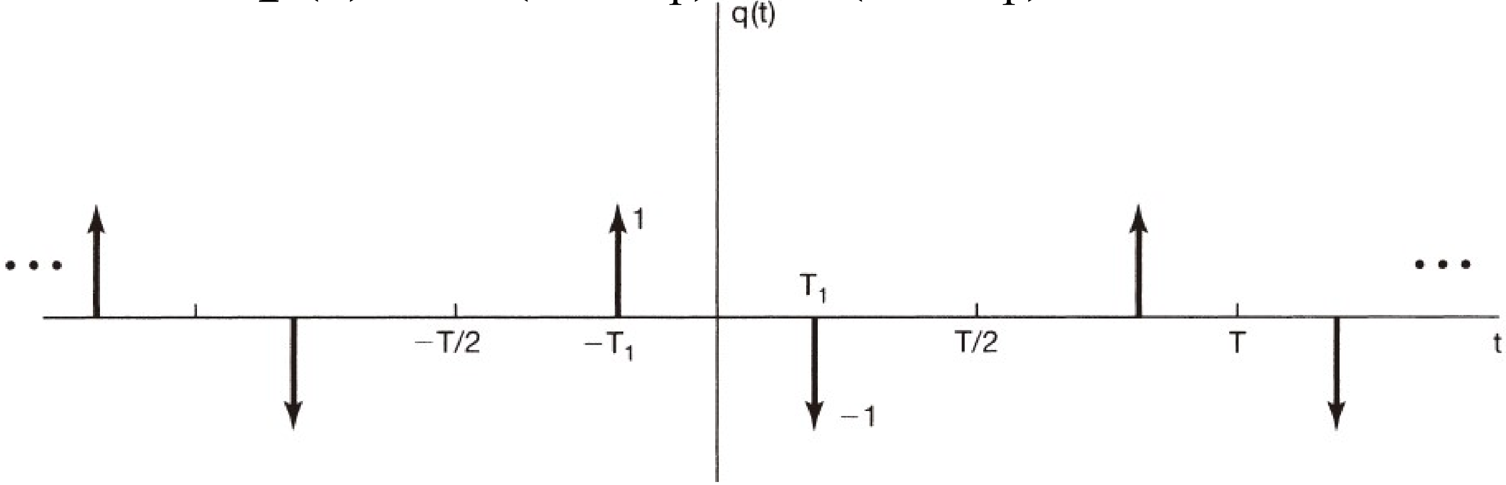
*constant (independent of k)

The impulse train also has a straightforward relationship to square-wave signals such as *g(t).*



The derivative of *g(t)* is the *q(t)* as the difference of two shifted versions of the impulse train *x(t).*

$$q(t) = x(t + T_1) - x(t - T_1). \qquad (3.77)$$



$$q(t) = x(t + T_1) - x(t - T_1).$$

$$x(t - t_0) \overset{\mathfrak{I}S}{\longleftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk(2\pi/T)t_0} a_k.$$

Using the properties of Fourier series, we can now compute the Fourier series coefficients of *q(t)* and *g(t)* without any further direct evaluation of the Fourier series analysis equation. First, from the time-shifting and linearity properties, the Fourier series coefficients b_k of *q(t)* may be expressed in terms of the Fourier series coefficients a_k of *x(t)*; that is,

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k, \qquad \omega_0 = 2\pi / T$$

Example 3.8

$$a_k = \frac{1}{T}(3.76)$$

Using eq. (3.76), we then have

$$b_k = \frac{1}{T}\left[e^{jk\omega_0 T_1} - e^{-jk\omega_0 T1}\right] = \frac{2j\sin(k\omega_0 T_1)}{T}.$$

Finally, since *q(t)* is the derivative of *g(t),* we can use the differentiation property in Table 3.1 to write

$$b_k = jk\omega_0 c_k, \qquad (3.78)$$

where the c_k are the Fourier series coefficients of *g(t)*. Thus,

$$c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, \qquad k \neq 0, \qquad (3.79)$$

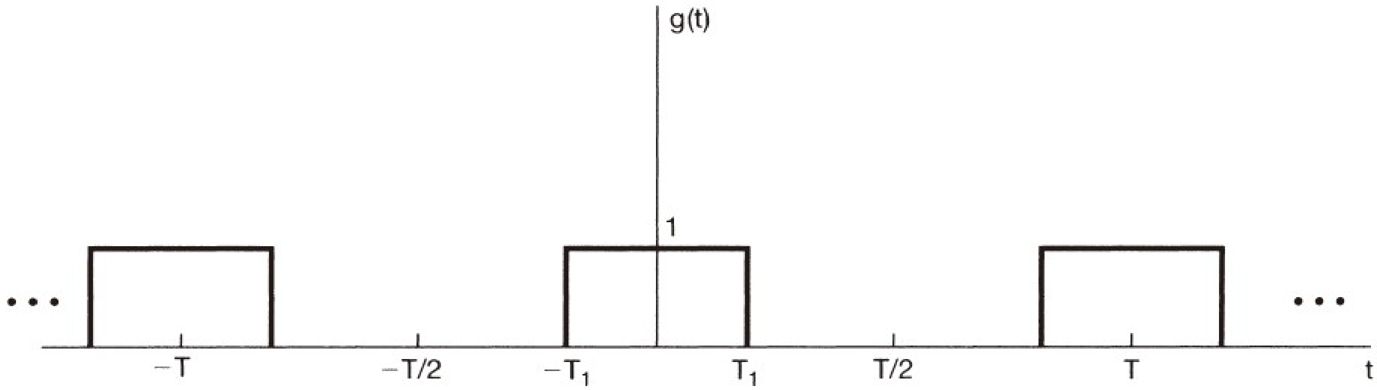
$$\omega_0 = 2\pi / T$$

Example 3.8

$$c_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, \qquad k \neq 0 \quad (3.79)$$

Note that eq. (3.79) is valid for $k \neq 0$, since we cannot solve for c_0 from eq. (3.78) with $k = 0$. However, since c_0 is just the average value of *g(t)* over one period, we can determine it by inspection from Figure 3.12(b): $c_0 = \frac{2T_1}{T}$.

*You can also double check the Fourier series coefficients of the square wave derived in Example 3.5.



3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

As defined in Chapter 1, a discrete-time signal *x[n]* is periodic with period *N* if

$$x[n] = x[n+N]. \qquad (3.84)$$

the set of all discrete-time complex exponential signals that are periodic with period N is given by

$$\phi_k[n] = e^{jw\omega_0 n} = e^{jk(2\pi/N)n}, k = 0, \pm 1, \pm 2, \ldots . \qquad (3.85)$$

All of these signals have fundamental frequencies that are multiples of $2\pi/N$ and thus are harmonically related.

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

$$\varphi_k[n] = e^{jk(2\pi/N)n}$$

This is a consequence of the fact that discrete-time complex exponentials which differ in frequency by a multiple of 2π are identical. Specifically,

$\phi_0[n] = \phi_N[n], \phi_1[n] = \phi_{N+1}[n]$, and, in general,

$$\phi_k[n] = \phi_{k+rN}[n]. \tag{3.86}$$

3.6.1 Linear Combinations of Harmonically Related Complex Exponentials

Such a linear combination has the form

$$x[n] = \sum_{k} a_k \varphi_k[n] = \sum_{k} a_k e^{jk\omega_0 n} = \sum_{k} a_k e^{jk(2\pi/N)n}. \qquad (3.87)$$

$$\varphi_k[n] = \varphi_{k+rN}[n].(3.86)$$

$$x[n] = \sum_{k=\langle N\rangle} a_k \varphi_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}. \qquad (3.88)$$

$$\langle N\rangle = 0, 1, 2, \ldots, (N-1) or 2, 3, \ldots, (N+1)$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

If we evaluate eq. (3.88) for N successive values of n corresponding to one period of x[n], we obtain

$$
\begin{aligned}
x[0] &= \sum_{k=\langle N\rangle} a_k, \\
x[1] &= \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N}, \\
&\vdots \\
x[N-1] &= \sum_{k=\langle N\rangle} a_k e^{j2\pi k(N-1)/N}.
\end{aligned}
\qquad (3.89)
$$

We can solve for $\{a_k\}$ by given these linearly equations, but these is an easy way.

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

$$x[n]=\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \quad (3.88)$$

We know that(shown in Problem 3.54)

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n} = \begin{cases} N, & k=0,\pm N,\pm 2N,.... \\ 0, & otherwise \end{cases} \quad (3.90)$$

Now consider the Fourier series representation of eq. (3.88). Multiplying both sides by $e^{-jr(2\pi/N)n}$ and summing over N terms, we obtain

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{j(k-r)(2\pi/N)n} \quad (3.91)$$

Interchanging the order of summation on the right-hand side, we have

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} \sum_{k=\langle k\rangle} a_k \sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n} \quad (3.92)$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

The right-hand side of eq. (3.92) then reduces to $a_r N$, and we have

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = a_r N \Rightarrow a_r = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n}. \quad (3.93)$$

This provides a closed-form expression for obtaining the Fourier series coefficients, and we have the discrete-time Fourier series pair:

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}, \quad (3.94)$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk\omega_0 n} = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}. \quad (3.95)$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

Referring to eq. (3.88), we see that if we take k in the range from 0 to $N-1$, we have

$$x[n] = a_0\phi_0[n] + a_1\phi_1[n] + \ldots + a_{N-1}\phi_{N-1}[n]. \quad (3.96)$$

Similarly, if k ranges from 1 to N, we obtain

$$x[n] = a_1\phi_1[n] + a_2\phi_2[n] + \ldots + a_N\phi_N[n]. \quad (3.97)$$

$$\phi_k[n] = \phi_{k+rN}[n]. \quad (3.86)$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

By letting k range over any set of N consecutive integers and using eq. (3.86),

$$a_k = a_{k+N}. \tag{3.98}$$

That is, if we consider more than N sequential values of k, the values a_k repeat periodically with period N.

Example 3.10

Consider the signal

$$x[n] = \sin \omega_0 n, \tag{3.99}$$

Which is the discrete-time counterpart of the signal x(t)= $\sin \omega_0 t$ of Example 3.3. *x[n]* is periodic only if $2\pi/\omega_0$ is an integer or a ratio of integers For the case when $2\pi/\omega_0$ is an integer *N*, that is, when

$$\omega_0 = \frac{2\pi}{N},$$

Example 3.10

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n} \quad (3.94)$$

x[n] is periodic with fundamental period N, and we obtain a result that is exactly analogous to the continuous-time case. Expanding the signal as a sum of two complex exponenitals, we get

$$x[n] = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}. \quad (3.100)$$

Comparing eq. (3.100) with eq.(3.94), we see by inspection that

$$a_1 = \frac{1}{2j}, \qquad a_{-1} = -\frac{1}{2j}, \quad (3.101)$$

Example 3.10

And the remaining coefficients over the interval of summation are zero. As described previously, these coefficients repeat with period N; thus, a_{N+1} is also equal to (1/2*j*) and a_{N-1} equals (-1/2*j*). The Fourier series coefficients for this example with $N = 5$ are illustrated in Figure 3.13.

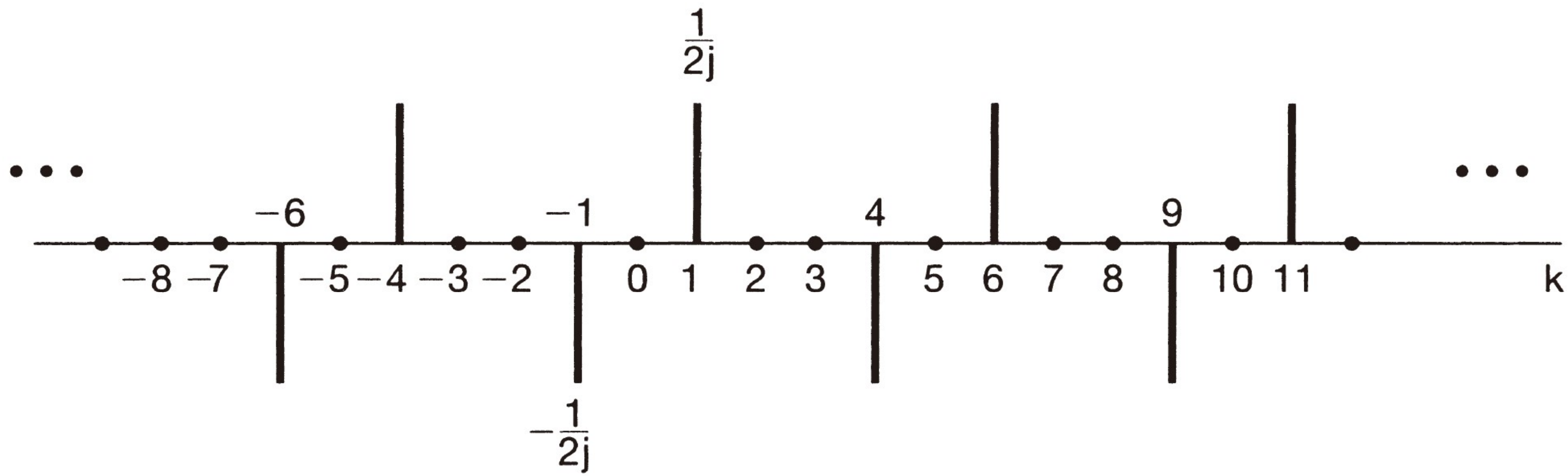


Figure 3.13 Fourier coefficients for $x[n] = \sin(2\pi/5)n$.

Example 3.10

Consider now the case when $2\pi/\omega_0$ is a ratio of integers—that is , when

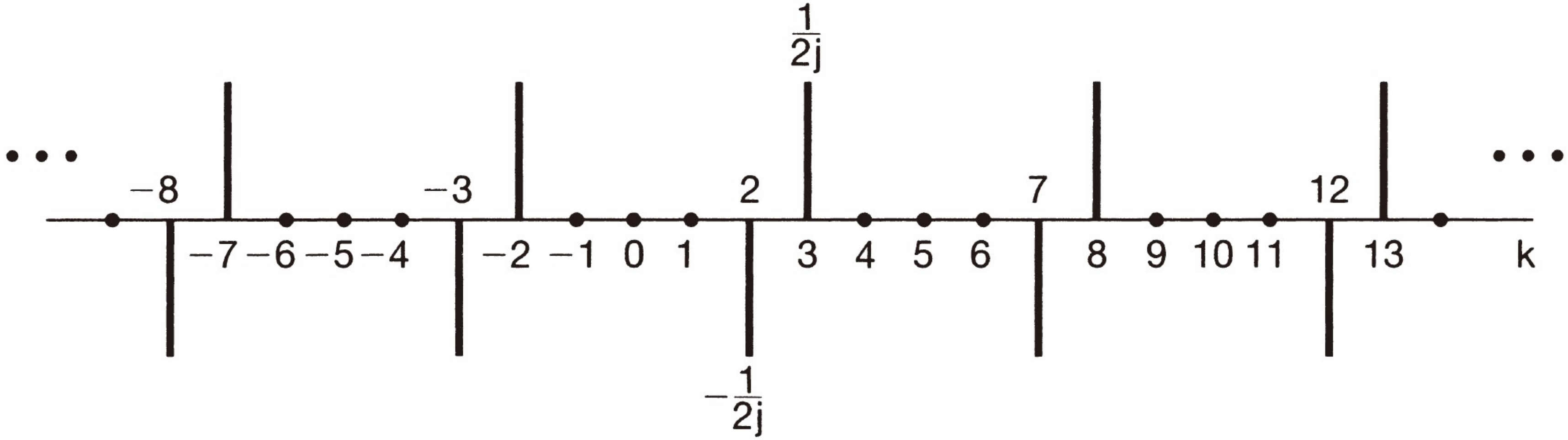
$$\omega_0 = \frac{2\pi M}{N}.$$

Assuming that M<N, and *M* and *N* do not have any common factors, *x[n]* has fundamental period of *N*. Again expanding *x[n]* as a sum of two complex exponentials, we have

$$x[n] = \frac{1}{2j} e^{jM(2\pi/N)n} - \frac{1}{2j} e^{-jM(2\pi/N)n},$$

Example 3.10

From which we can determine by inspection that $a_M = (1/2j)$, $a_{-M} = (-1/2j)$, and the remaining coefficients over one period of length N are zero. The Fourier coefficients for this example with $M = 3$ and $N = 5$ are in Figure 3.14. Again, we have indicated the periodicity of the coefficients. For example, for $N = 5$, $a_2 = a_{-3}$, which in our example equals (-1/2j).



3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

- Convergence

Assume that the period N is odd. In Figure 3.18, we have depicted the signals

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n} \quad (3.106)$$

approximating example of Figure 3.16 with $N = 9$, $2N_1 + 1 = 5$, and for several values of M. For $M = 4$, the partial sum **exactly** equals *x[n]*.

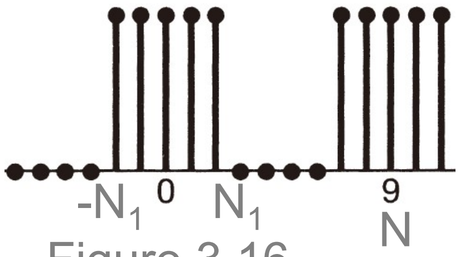
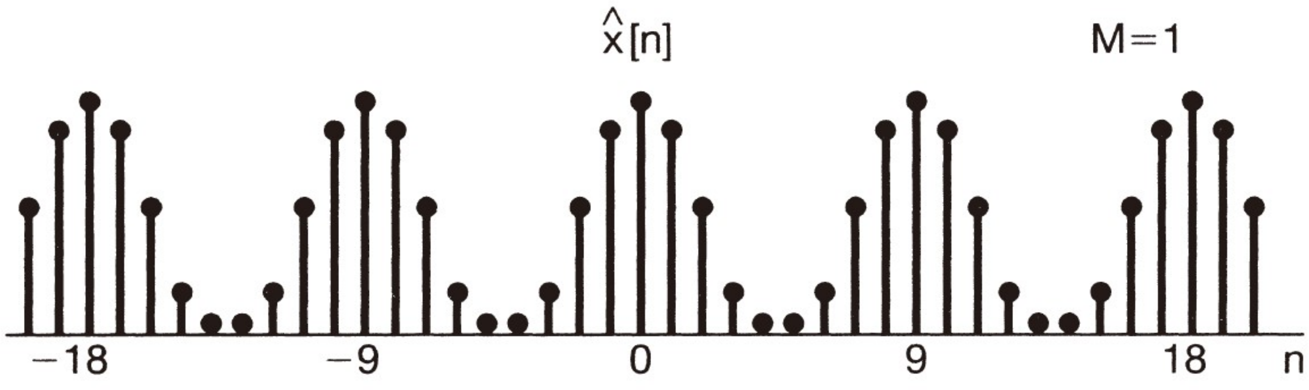


Figure 3.16

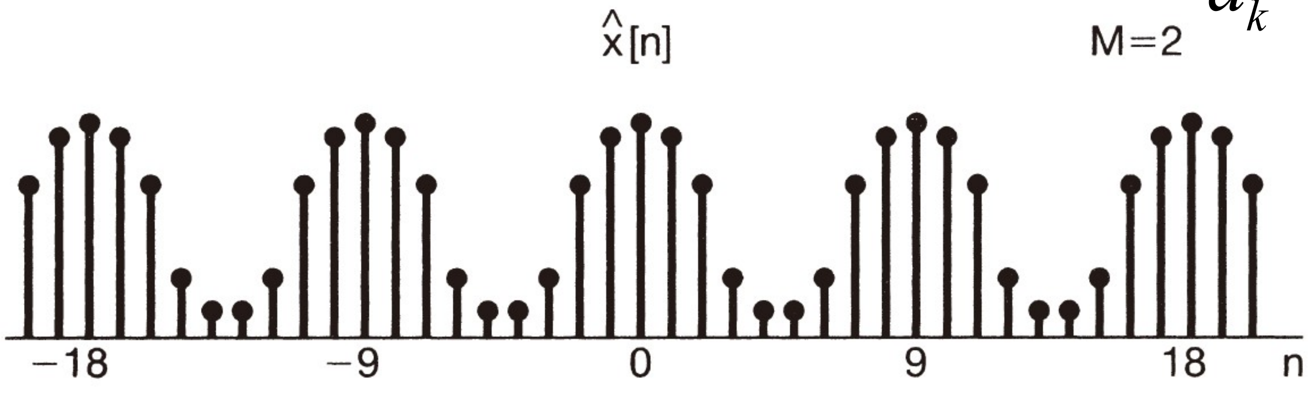
$$a_k = \begin{cases} \dfrac{1}{N}\dfrac{\sin(2\pi k(N_1+1/2)/N)}{\sin(\pi k/N)}, & for \quad k \neq 0, \pm N, \pm 2N, \ldots \\ \dfrac{2N_1+1}{N}, & for \quad k = 0, \pm N, \pm 2N, \ldots \end{cases}.$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n}$$



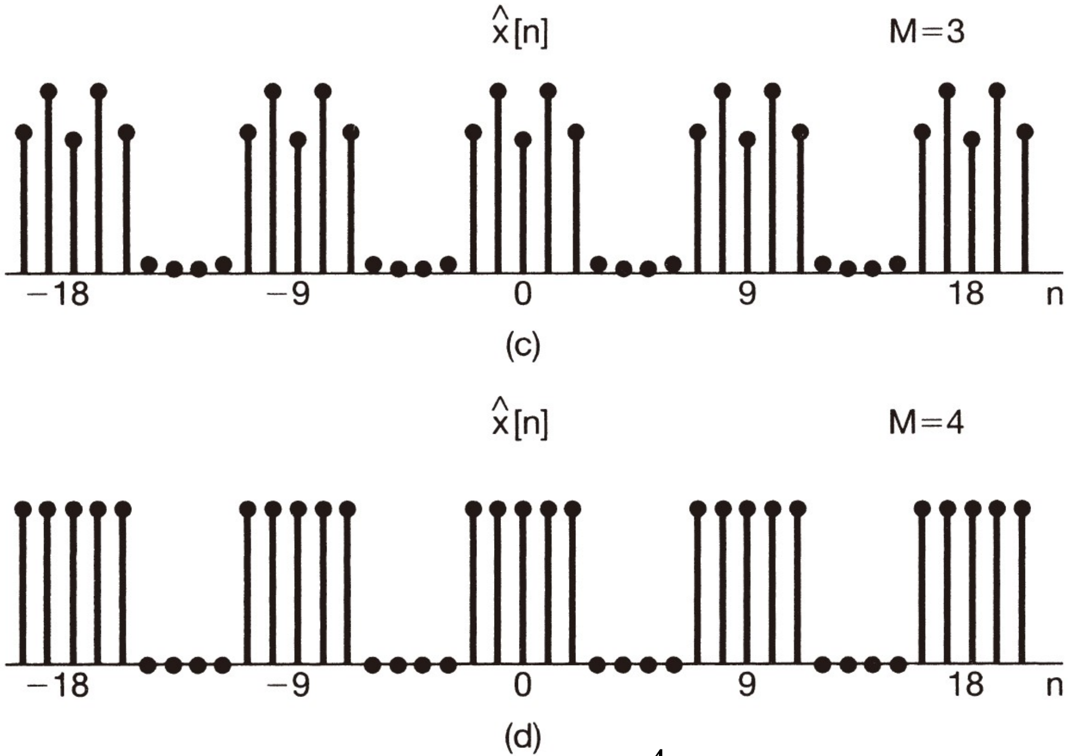
(a)



(b)

$$a_k = \begin{cases} \frac{1}{N}\frac{\sin(2\pi k(N_1+1/2)/N)}{\sin(\pi k/N)}, & \text{for } k \neq 0, \pm N, \pm 2N, \ldots \\ \frac{2N_1+1}{N}, & \text{for } k = 0, \pm N, \pm 2N, \ldots \end{cases}$$

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal



$$x[n] = \hat{x}[n] = \sum_{k=-4}^{4} a_k e^{jk(2\pi/N)n}$$

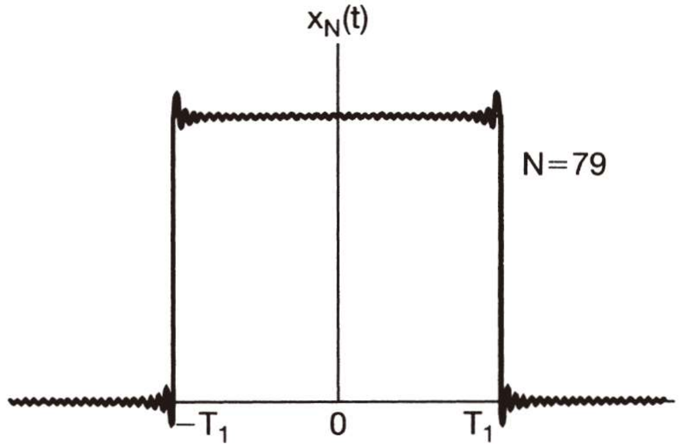


Figure 3.18 Partial sums of eqs. (3.106) and (3.107) for the periodic square wave of Figure 3.16 with $N = 9$ and $2N_1 + 1 = 5$: (a) $M = 1$; (b) $M = 2$; (c) $M = 3$; (d) $M = 4$.

3.6.2 Determination of the Fourier Series Representation of a Periodic Signal

If N is odd and we take $M = (N - 1)/2$ in eq. (3.106), the sum includes exactly N terms, and consequently, from the synthesis equations, we have $\tilde{x}[n] = x[n]$. Similarly, if N is even and we let

$$\tilde{x}[n] = \sum_{k=-M+1}^{M} a_k e^{jk(2\pi/N)n},$$

3.7 Properties of Discrete-Time Fourier Series

TABLE 3.2 PROPERTIES OF DISCRETE-TIME FOURIER SERIES 表 3.2 離散時間傅立葉級數的性質

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n] \end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k \end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n - n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n} x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m} a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N \rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N \rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1 - e^{-jk(2\pi/N)})}\right) a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{Ev}\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{Od}\{x[n]\} & [x[n] \text{ real}] \end{cases}$	$\begin{array}{l} \mathcal{Re}\{a_k\} \\ j\mathcal{Im}\{a_k\} \end{array}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N} \sum_{n=\langle N \rangle} |x[n]|^2 = \sum_{k=\langle N \rangle} |a_k|^2$$

3.7 Properties of Discrete-Time Fourier Series

If $x[n]$ is a periodic signal with period N and with Fourier series coefficients denoted by a_k , then we will write

$$x[n] \overset{FS}{\longleftrightarrow} a_k .$$

3.7.1 Multiplication

The product of two periodic signals with period N results in a periodic signal with period N whose sequence of Fourier series coefficients is the **periodic convolution** of the sequences of Fourier series coefficients of the two signals being multiplied.

$$x[n] \xleftrightarrow{FS} a_k \quad \text{and} \quad y[n] \xleftrightarrow{FS} b_k$$

This is shown in Problem 3.57, its Fourier coefficients, $\{d_k\}$, are given by

$$x[n]y[n] \xleftrightarrow{FS} d_k = \sum_{l=\langle N \rangle} a_l b_{k-l} \qquad (3.108)$$

3.7.1 Multiplication

Equation (3.108) is analogous to the definition of convolution, **except that the summation variable is now restricted to an interval of *N* consecutive samples.**

(3.108) $$x[n]y[n] \overset{FS}{\longleftrightarrow} d_k = \sum_{l=\langle N\rangle} a_l b_{k-l}$$

$$\sum_{r=\langle N\rangle} x[r]y[n-r] \overset{FS}{\longleftrightarrow} N a_k b_k \quad Periodic \quad convolution$$

3.7.2 First Difference

$$y[n] = x[n] - x[n-1]$$

If *x[n]* is periodic with period *N*, then so is *y[n],* since shifting *x[n]* or linearly combining *x[n]* with another periodic signal whose period is N always results in a periodic signal with period N. Also, if

$$x[n] \stackrel{FS}{\longleftrightarrow} a_k,$$

then the Fourier coefficients corresponding to the first difference of *x[n]* may be expressed as

$$x[n] - x[n-1] \stackrel{FS}{\longleftrightarrow} (1 - e^{-jk(2\pi/N)})a_k \qquad (3.109)$$

$$x[n-1] \stackrel{FS}{\longleftrightarrow} a_k e^{-jk(2\pi/N)}$$

3.7.3 Parseval's Relation for Discrete-Time Periodic Signals

As shown in Problem 3.57, Perseval's relation for discrete-time periodic signals is given by

$$\frac{1}{N}\sum_{n=\langle N\rangle}\left|x[n]\right|^2=\sum_{k=\langle N\rangle}\left|a_k\right|^2, \qquad (3.110)$$

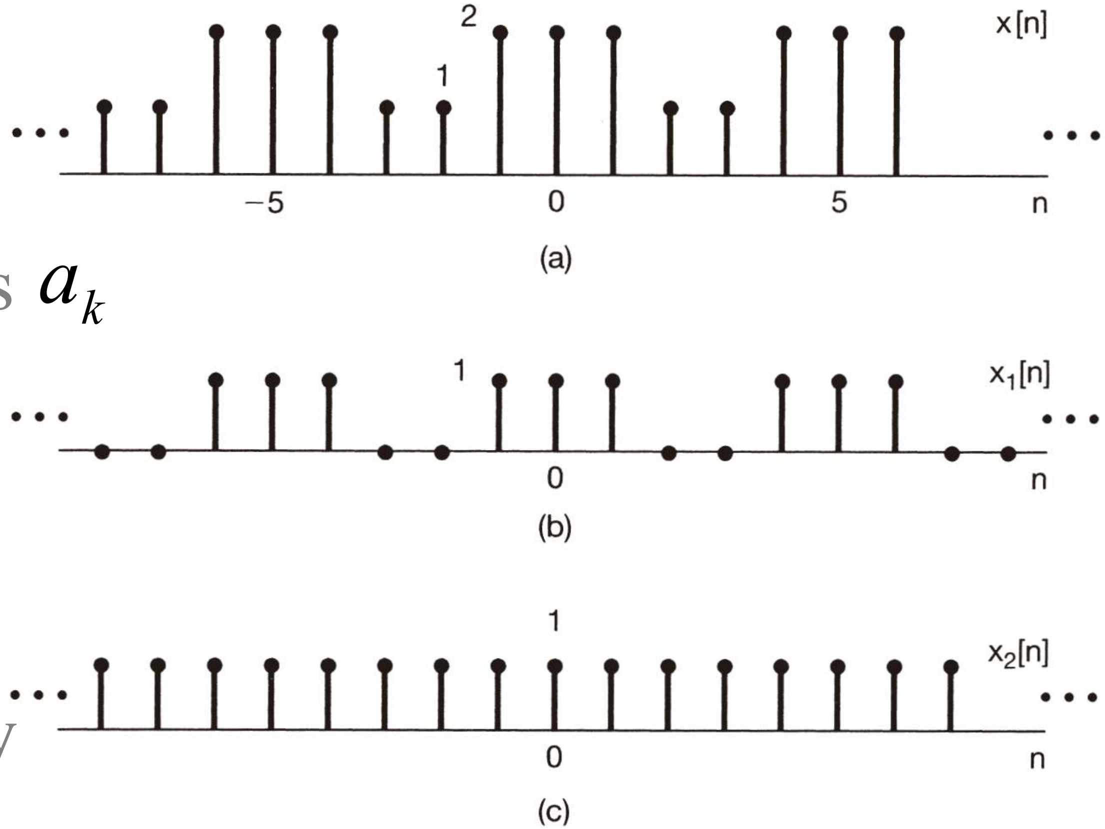
where the a_k are the Fourier series coefficients of *x[n]* of *N* is the period.

$$x[n]=\sum_{k=<N>} a_k e^{jk(2\pi/N)n}$$

$$\sum_{n=\langle N\rangle} e^{jk(2\pi/N)n}=\begin{cases} N, & k=0,\pm N,\pm 2N,.... \\ 0, & otherwise \end{cases}$$

Example 3.13

Let us consider the problem of finding the Fourier series coefficients a_k of the sequence *x[n]* shown in Figure 3.19(a). This sequence has a fundamental period of 5. We observe that *x[n]* may be viewed as the sum of the square wave $x_1[n]$ in Figure 3.19(b) and the dc sequence $x_2[n]$ in Figure 3.19(c).



Example 3.13

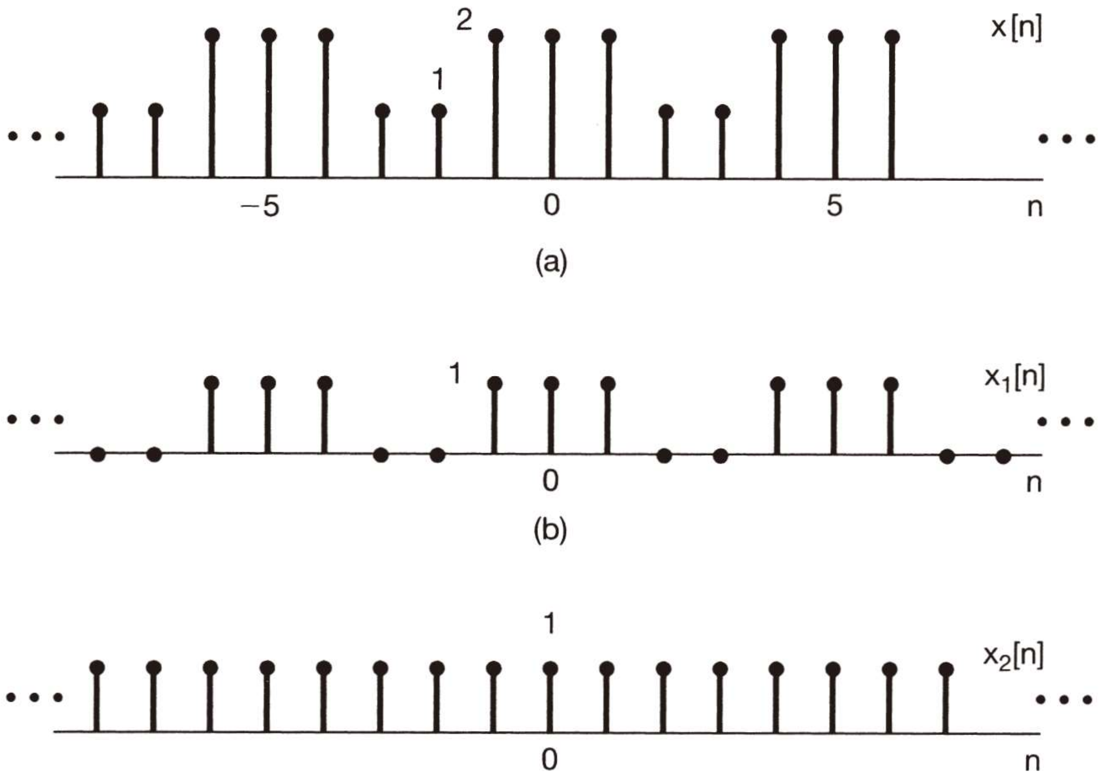
Denoting the Fourier series coefficients of $x_1[n]$ by b_k and those of $x_2[n]$ by c_k, we use the **linearity property of Table 3.2** to conclude that

$$a_k = b_k + c_k. \tag{3.111}$$

a_k

b_k

c_k



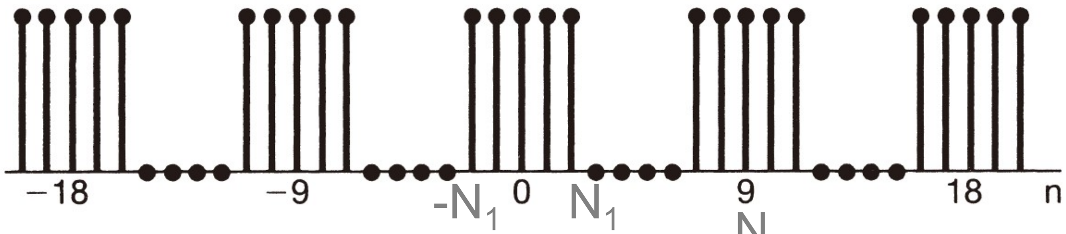


Figure 3.16

$$a_k = \begin{cases} \dfrac{1}{N}\dfrac{\sin(2\pi k(N_1+1/2)/N)}{\sin(\pi k/N)}, \\ \qquad for \;\; k \neq 0, \pm N, \pm 2N, \ldots \\ \dfrac{2N_1+1}{N}, \\ \qquad for \;\; k = 0, \pm N, \pm 2N, \ldots \end{cases}.$$

From Example 3.12 (with $N_1 = 1$ and $N = 5$), the Fourier series coefficients corresponding to $x_1[n]$ can be expressed as

$$b_k = \begin{cases} \dfrac{1}{5}\dfrac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & for \quad k \neq 0, \pm 5, \pm 10, \ldots \\ \dfrac{3}{5}, & for \quad k = 0, \pm 5, \pm 10, \ldots \end{cases}. \qquad (3.112)$$

The sequence $x_2[n]$ has **only** a dc value, which is captured by its zeroth Fourier series coefficient:

$$c_0 = \frac{1}{5}\sum_{n=0}^{4} x_2[n] = 1. \qquad (3.113)$$

Example 3.13

Since the discrete-time Fourier series coefficients are periodic, it follows that $c_k = 1$ whenever k is an integer multiple of 5. The remaining coefficients of $x_2[n]$ must be zero, because $x_2[n]$ contains only a dc component. We can now substitute the expressions for b_k and c_k into eq. (3.111) to obtain

$$a_k = \begin{cases} b_k = \frac{1}{5}\frac{\sin(3\pi k/5)}{\sin(\pi k/5)}, & for \quad k \neq 0, \pm 5, \pm 10, \ldots \\ \frac{8}{5}, & for \quad k = 0, \pm 5, \pm 10, \ldots \end{cases} \qquad (3.114)$$

3.8 Fourier Series and LTI Systems

In continuous time, if $x(t) = e^{st}$ is the input to a continuous-time LTI system, then the output is given by $y(t) = H(s)e^{st}$, where, from eq. (3.6),

$$H(s) = \int_{-\infty}^{+\infty} h(\tau) e^{-s\tau} d\tau, \qquad (3.119)$$

in which $h(t)$ is the impulse respose of the LTI system

3.8 Fourier Series and LTI Systems

Similarly, if $x[n] = z^n$ is the input to a discrete-time LTI system, then the output is given by $y[n] = H(z)z^n$, where, from eq. (3.10),

$$H(z) = \sum_{k=-\infty}^{+\infty} h[k] z^{-k}, \tag{3.120}$$

in which $h(n)$ is the impulse response of the LTI system

3.8 Fourier Series and LTI Systems

The system function of the form $s = j\omega$—i.e., $H(j\omega)$ viewed as a function of ω—is referred to as the ***frequency response*** of the system and is given by

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t) e^{-j\omega t} dt. \tag{3.121}$$

連續時間系統的頻率響應函數

3.8 Fourier Series and LTI Systems

Then the system function *H(z)* for *z* restricted to the form $z = e^{j\omega}$ is referred to as the **frequency response** of the system and is given by

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n] e^{-j\omega n}. \quad (3.122)$$

離散時間系統的頻率響應函數

3.8 Fourier Series and LTI Systems

$$x(t) = e^{st}$$

$$y(t) = H(s)e^{st}$$

Consider first the continuous-time case, and let *x(t)* be a periodic signal with a Fourier series representation given by

$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}. \qquad (3.123)$$

Suppose we apply x(t) as the input to an LTI system with impulse response h(t)

In eq. (3.13) with $s_k = jk\omega_0$, it follows that the output is

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}. \qquad (3.124)$$

3.8 Fourier Series and LTI Systems

$$y(t) = \sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}.$$

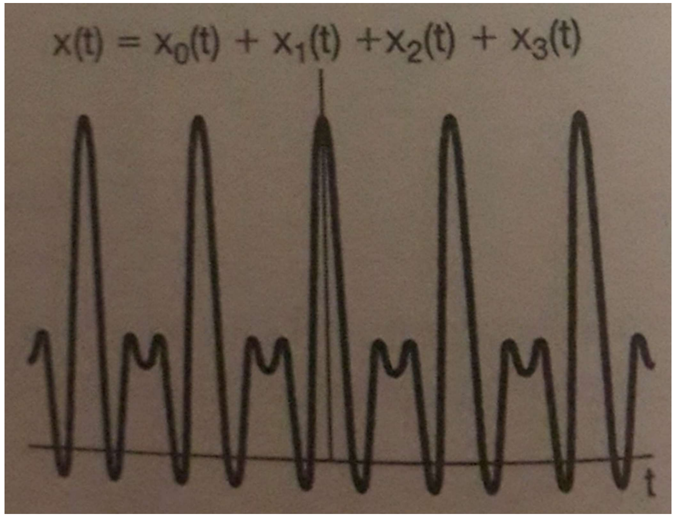
Thus, *y(t)* is also periodic with the same fundamental frequency as *x(t)*. Furthermore, if $\{a_k\}$ is the set of Fourier series coefficients for the input *x(t)*, then $\{a_k H(jk\omega_0)\}$ is the set of coefficients for the output *y(t)*.

Example 3.16

Suppose that the periodic signal *x(t)* discussed in Example 3.2 is the input signal to an LTI system with impulse response

$$h(t) = e^{-t}u(t).$$

$$x(t) = \sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$



Example 3.16

$$h(t) = e^{-t}u(t).$$

$$H(j\omega) = \int_{-\infty}^{+\infty} h(t)e^{-j\omega t}\,dt.$$

To calculate the Fourier series coefficients of the output *y(t)*, we first compute the frequency response:

$$H(j\omega) = \int_{0}^{\infty} e^{-\tau}e^{-j\omega\tau}\,d\tau$$

$$= -\frac{1}{1+j\omega}e^{-\tau}e^{-j\omega\tau}\Big|_{0}^{\infty} \qquad (3.125)$$

$$= \frac{1}{1+j\omega}.$$

Example 3.16

$$y(t)=\sum_{k=-\infty}^{+\infty} a_k H(jk\omega_0)e^{jk\omega_0 t} \quad (3.124)$$

$$H(j\omega_0)=1/(1+j\omega) \quad (3.125)$$

$$x(t)=\sum_{k=-3}^{+3} a_k e^{jk2\pi t}$$

Therefore, using eqs. (3.124) and (3.125), together with the fact that $\omega_0 = 2\pi$ in this example, we obtain

$$y(t)=\sum_{k=-3}^{+3} b_k e^{jk2\pi t}, \quad (3.126)$$

where $b_k = a_k H(jk2\pi)$

Example 3.16

$$H(j\omega_0) = 1/(1+j\omega)$$

(3.125)

$$a_0 = 1,$$

$$a_1 = a_{-1} = \frac{1}{4},$$

$$a_2 = a_{-2} = \frac{1}{2},$$

$$a_3 = a_{-3} = \frac{1}{3}.$$

with $b_k = a_k H(jk2\pi)$, so that

$$b_0 = 1,$$

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right), \qquad b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right),$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right), \qquad b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right),$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right), \qquad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right).$$

(3.127)

$$a_k = A_k e^{j\theta_k}, \qquad x(t) = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k) \quad (3.31)$$

$$a_k = B_k + jC_k, \qquad x(t) = a_0 + 2\sum_{k=1}^{\infty} [B_k \cos k\omega_0 t - C_k \sin k\omega_0 t] \quad (3.32)$$

Note that *y(t)* must be a real-valued signal, since it is the convolution of *x(t)* and *h(t),* which are both real. This can be verified by examining eq. (3.127) and observing that $b_k^* = b_{-k}$. Therefore, *y(t)* can also be expressed in either of the forms given in eqs. (3.31) and (3.31);that is,

$$y(t) = 1 + 2\sum_{k=1}^{3} D_k \cos(2\pi kt + \theta_k), \qquad (3.128)$$

or

$$y(t) = 1 + 2\sum_{k=1}^{3} [E_k \cos 2\pi kt - F_k \sin 2\pi kt], \qquad (3.129)$$

Example 3.16

$$b_1 = \frac{1}{4}\left(\frac{1}{1+j2\pi}\right), \qquad b_{-1} = \frac{1}{4}\left(\frac{1}{1-j2\pi}\right),$$

$$b_2 = \frac{1}{2}\left(\frac{1}{1+j4\pi}\right), \qquad b_{-2} = \frac{1}{2}\left(\frac{1}{1-j4\pi}\right),$$

$$b_3 = \frac{1}{3}\left(\frac{1}{1+j6\pi}\right), \qquad b_{-3} = \frac{1}{3}\left(\frac{1}{1-j6\pi}\right).$$

where

$$b_k = D_k e^{j\theta_k} = E_k + jF_k, \qquad k = 1,2,3. \qquad (3.130)$$

These coefficients can be evaluated directly from eq.(3.127). For example,

$$D_1 = |b_1| = \frac{1}{4\sqrt{1+4\pi^2}}, \qquad \theta_1 = \measuredangle b_1 = -\tan^{-1}(2\pi),$$

$$E_1 = \Re e\{b_1\} = \frac{1}{4(1+4\pi^2)}, \qquad F_1 = \Im m\{b_1\} = -\frac{\pi}{2(1+4\pi^2)}.$$

3.8 Fourier Series and LTI Systems

$$y[n] = \sum_k a_k H(z_k) z_k^n \quad (3.16)$$

$$x[n] = z^n$$

$$y[n] = H(z) z^n$$

let *x[n]* be a periodic signal with Fourier series representation given by

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}.$$

If we apply this signal as the input to an LTI system with impulse response *h[n]*, then, as in eq. (3.16) with $z_k = e^{jk(2\pi/N)}$, the output is

$$y[n] = \sum_{k=\langle N \rangle} a_k H(e^{j2\pi k/N}) e^{jk(2\pi/N)n}.$$

3.9 Filtering

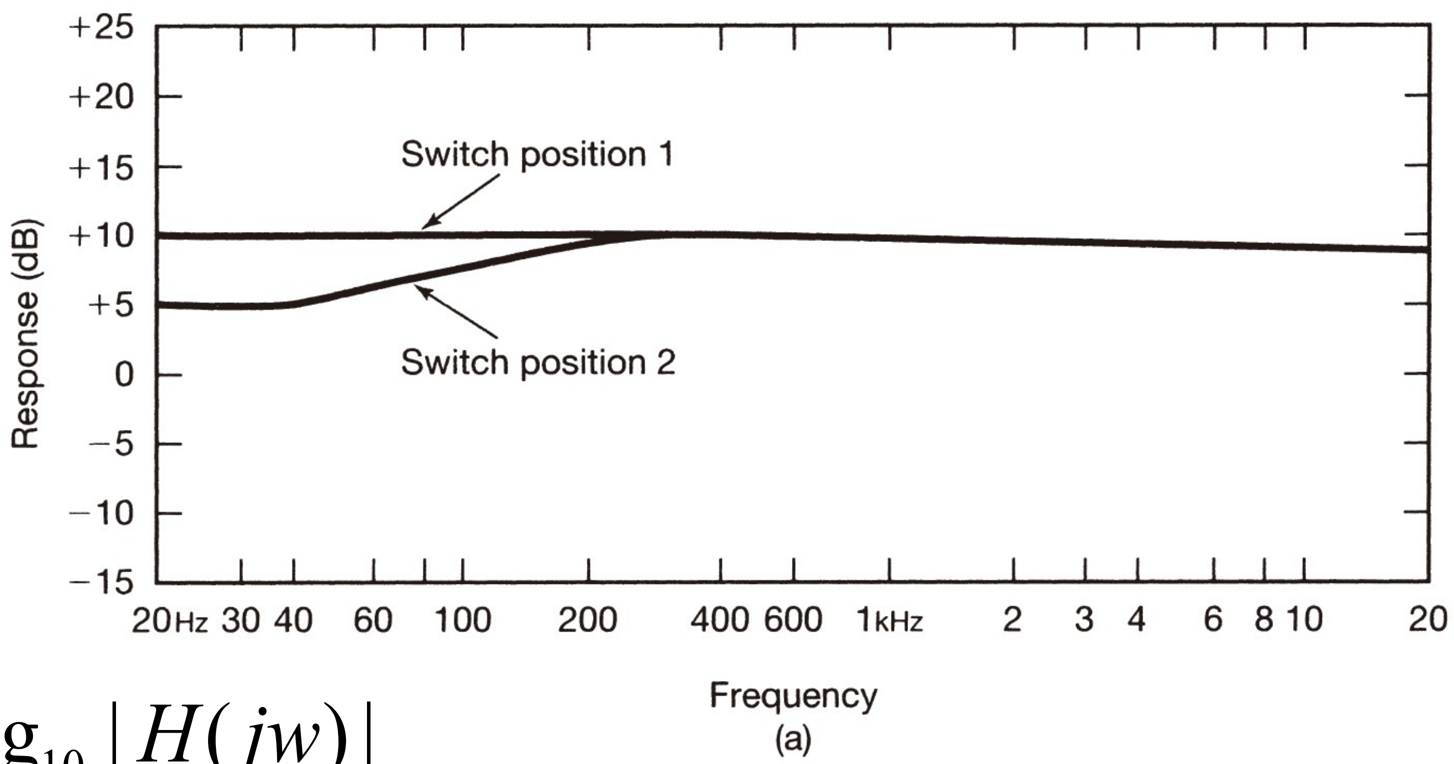
(Linear) **Filtering** can be conveniently accomplished through **the use of LTI systems** with an appropriately chose **frequency response**, and frequency-domain methods provide us with the ideal tools to examine this very important class of applications. In this and the following two sections, we take a first look at filtering through a few examples.

「濾波」即在對於訊號中某些頻率分量改變振幅或消除。用以改變訊號的頻譜形狀的LTI系統稱為「頻率整形濾波器」。用以在不失真之外通過某些頻率，或大大地縮減或消除其它頻率成分的，稱為「頻率選擇濾波器」。

3.9.1 Frequency-Shaping Filters

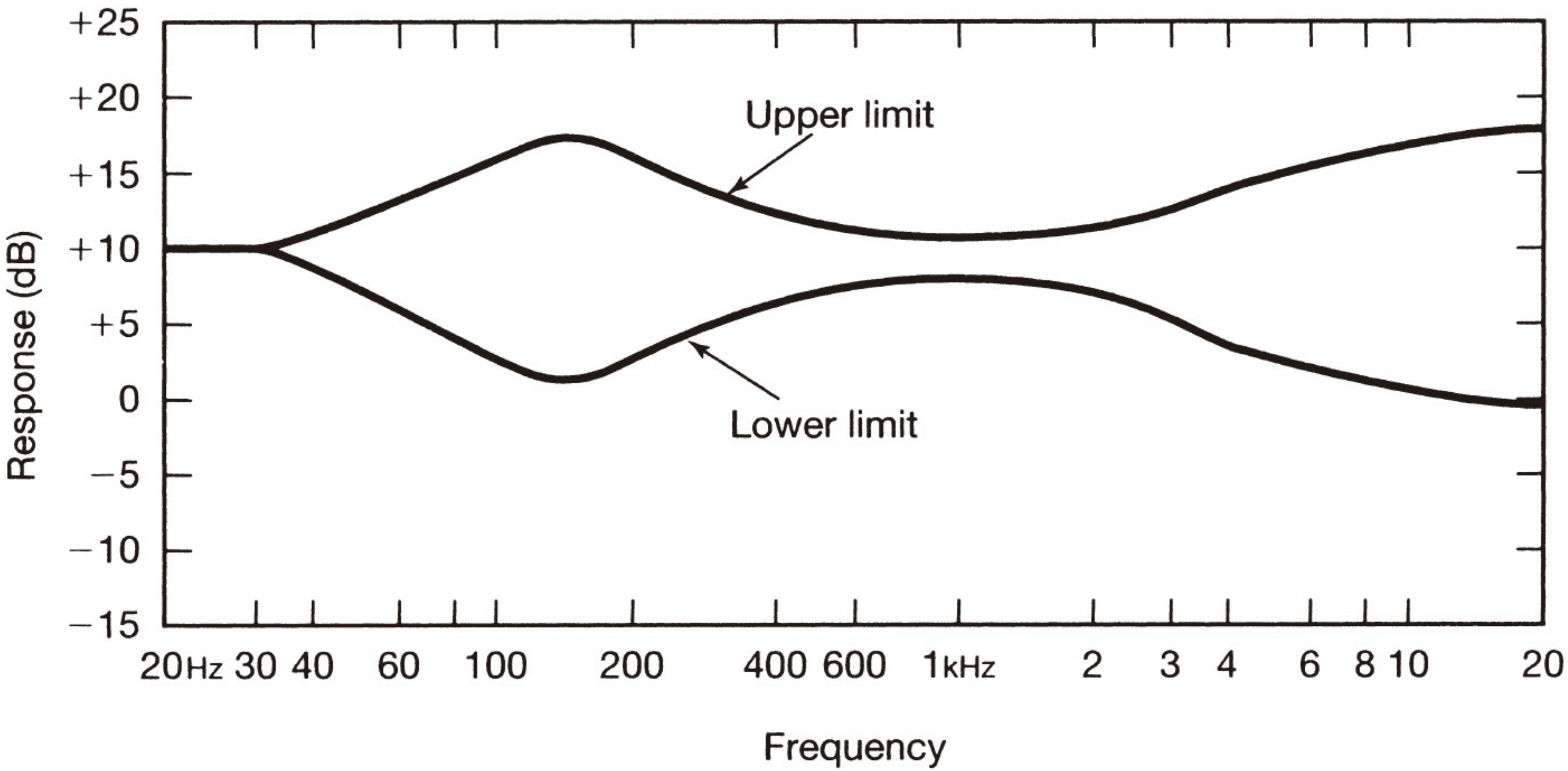
One application in which frequency-shaping filters are often encountered is audio systems.

頻率整形濾波器常應用於音訊系統。



$$20\log_{10}|H(jw)|$$

3.9.1 Frequency-Shaping Filters



(b)

3.9.1 Frequency-Shaping Filters

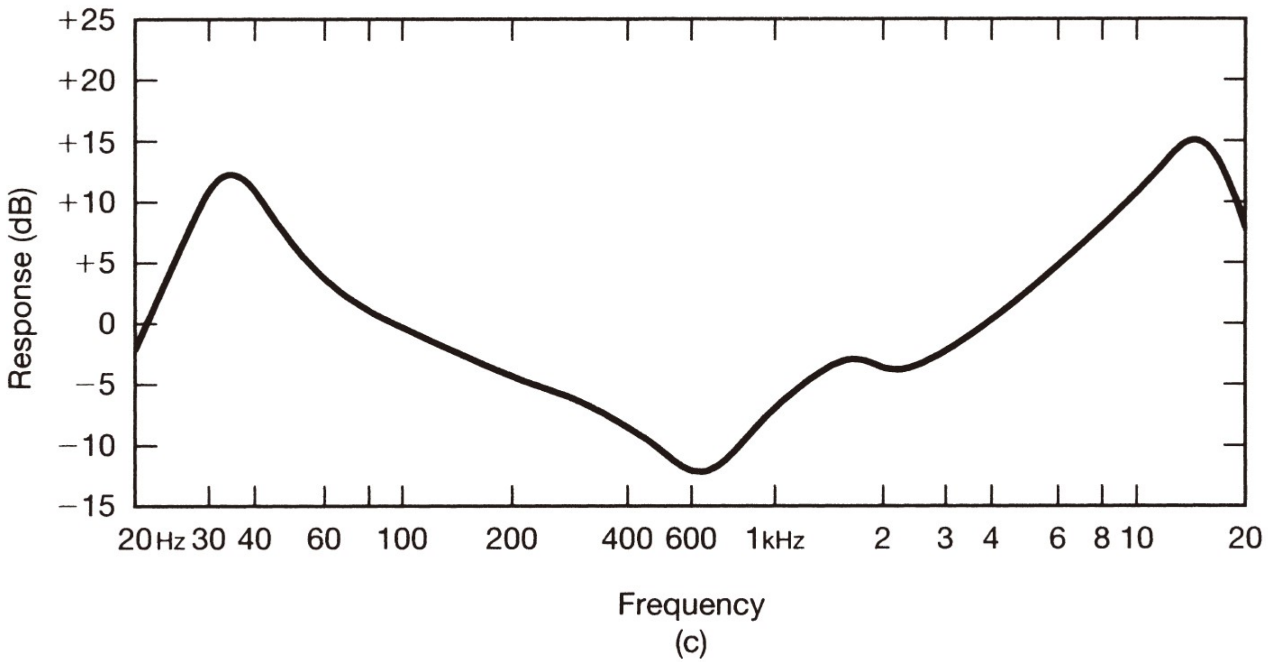
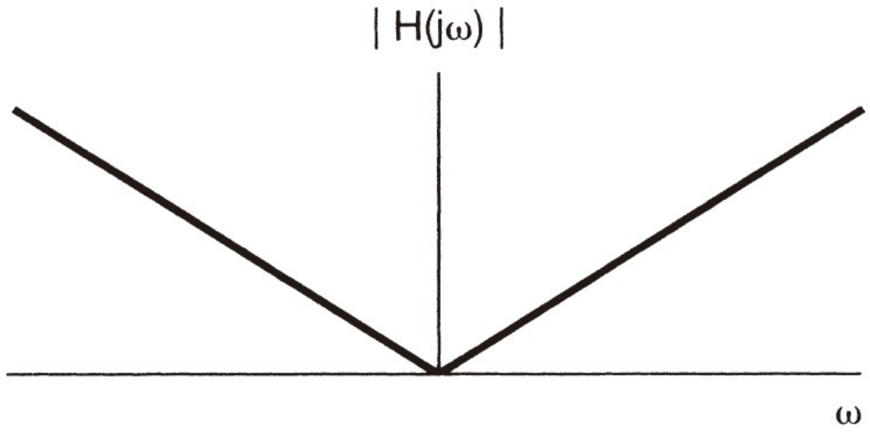


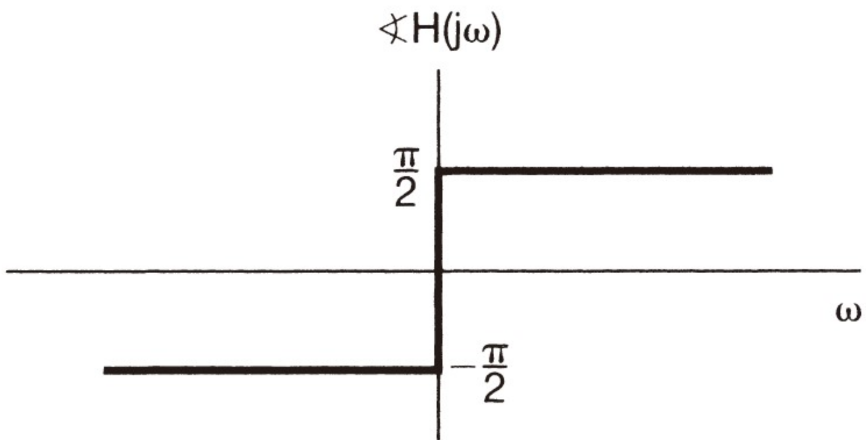
Figure 3.22 Magnitudes of the frequency responses of the equalizer circuits for one particular series of audio speakers, shown on a scale of $20 \log_{10} |H(j\omega)|$, which is referred to as a decibel (or dB) scale. (a) Low-frequency filter controlled by a two-position switch; (b) upper and lower frequency limits on a continuously adjustable shaping filter; (c) fixed frequency response of the equalizer stage.

3.9.1 Frequency-Shaping Filters

Another class of frequency-shaping filters often encountered is that for which the filter output is the derivative of the filter input, i.e., $y(t) = dx(t)/dt$. With $x(t)$ of the form $x(t) = e^{j\omega t}$, $y(t)$ will be $y(t) = j\omega e^{j\omega t}$, from which it follows that the frequency response is

$$H(j\omega) = j\omega. \qquad (3.137)$$





3.9.1 Frequency-Shaping Filters

$$First \quad difference: \ h[n] = \frac{1}{2}\left(\delta[n] - \delta[n-1]\right)$$

(a)

(b)

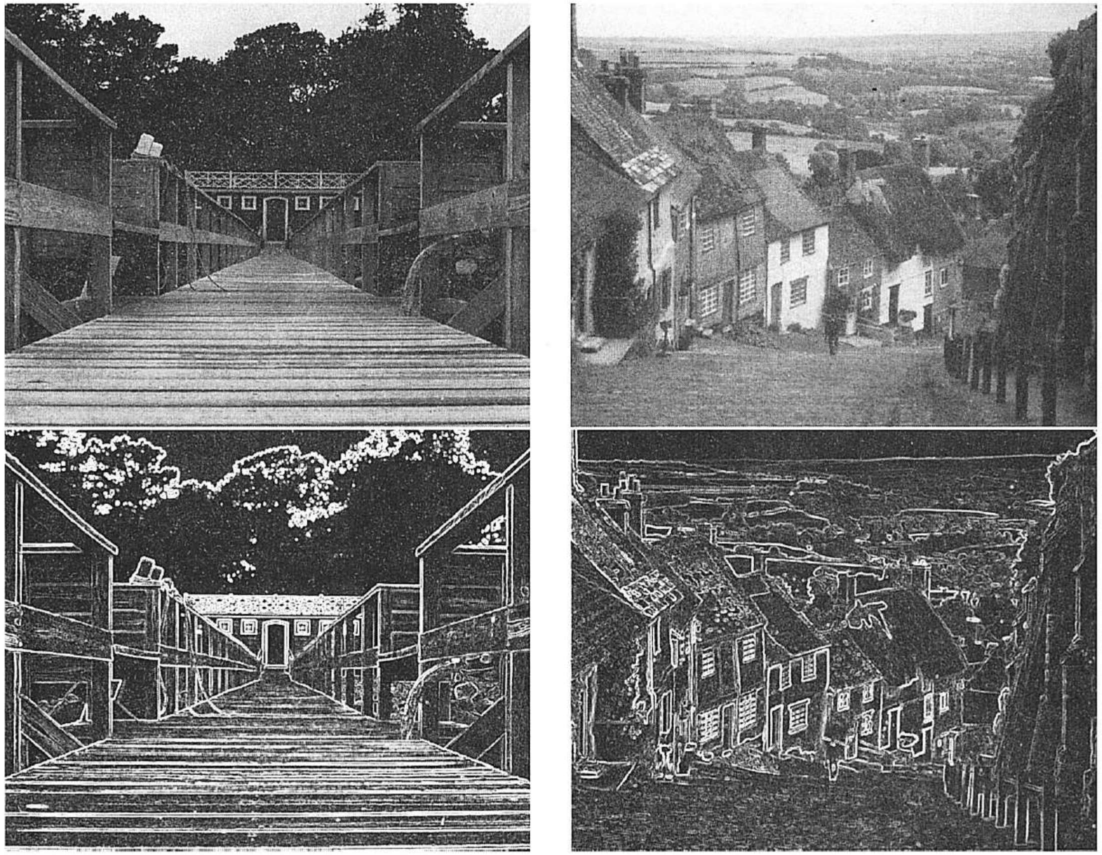


Figure 3.24 Effect of a differentiating filter on an image: (a) two original images; (b) the result of processing the original images with a differentiating filter.

3.9.1 Frequency-Shaping Filters

As one example of a simple discrete-time filter, consider an LTI system that successively takes a two-point average of the input values:

$$y[n] = \frac{1}{2}\left(x[n] + x[n-1]\right). \tag{3.138}$$

3.9.1 Frequency-Shaping Filters

In the case $h[n]=\frac{1}{2}(\delta[n]+\delta[n-1])$, and from eq. (3.122), we see that the frequency response of the system is

$$H(e^{j\omega})=\frac{1}{2}\left[1+e^{-j\omega}\right]=e^{-j\omega/2}\cos(\omega/2). \qquad (3.139)$$

$$H(e^{j\omega})=\sum_{n=-\infty}^{+\infty}h[n]e^{-j\omega n} \quad (3.122)$$

3.9.1 Frequency-Shaping Filters

if the input to this system is constant—i.e., a zero-frequency complex exponential $x[n] = Ke^{j0 \cdot n} = K$ –then the output will be

$$y[n] = H(e^{j \cdot 0})Ke^{j\omega 0 \cdot n} = K = x[n].$$

On the other hand, if the input is the high-frequency signal $x[n] = Ke^{j\pi n} = K(-1)^n$, then the output will be

$$y[n] = H(e^{j\pi})Ke^{j\pi \cdot n} = 0.$$

3.9.1 Frequency-Shaping Filters

$$H(e^{j\omega}) = e^{-j\omega/2}\cos(\omega/2).$$

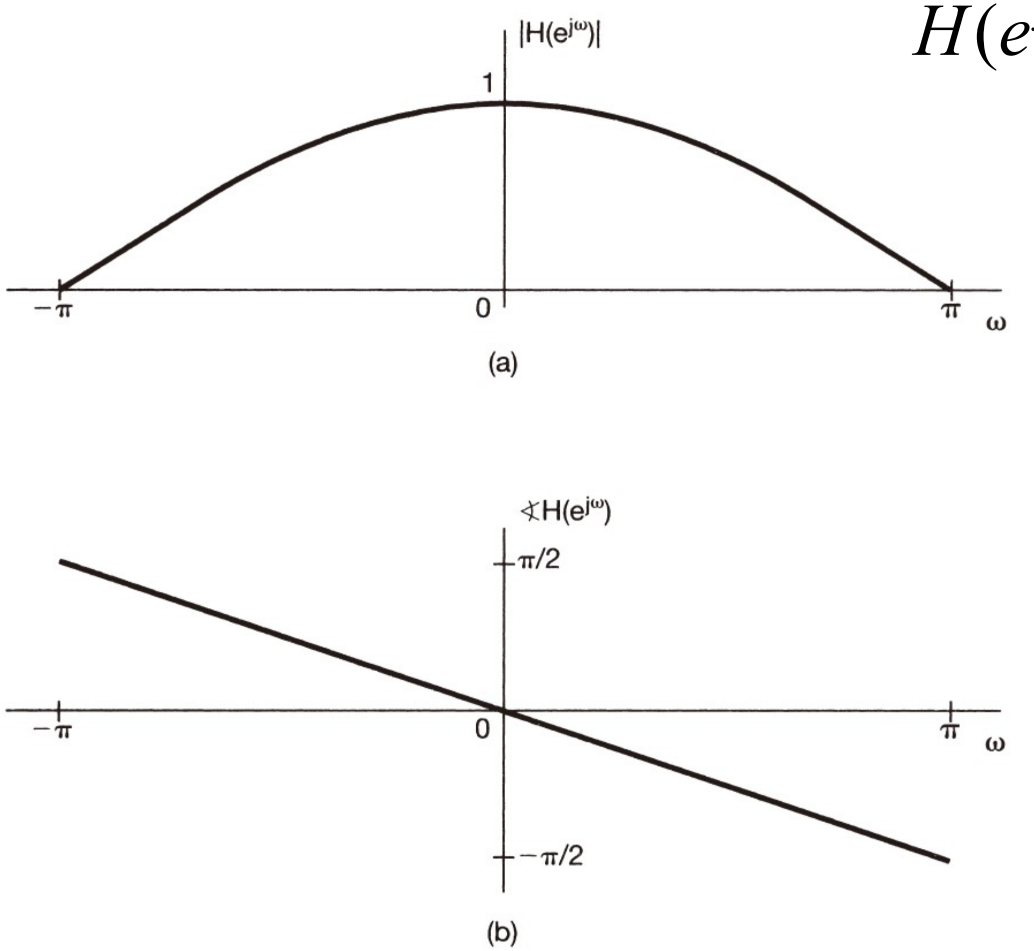


Figure 3.25 (a) Magnitude and (b) phase for the frequency response of the discrete-time LTI system $y[n] = 1/2(x[n] + x[n-1])$.

3.9.2 Frequency-Selective Filters

Frequency-selective filters are a class of filters specifically intended to **accurately or approximately select some bands** of frequencies and reject others. It is used a lot in systems such as communication systems.

頻率選擇濾波器常用於雜訊消除、通訊系統(如AM等)。

3.9.2 Frequency-Selective Filters

- A *lowpass filter* is a filter that passes low frequencies—i.e., frequencies around $\omega = 0$—and attenuates or rejects higher frequencies.
- A *highpass filter* is a filter that passes high frequencies and attentuates or rejects low ones.
- A *bandpass filter* is a filter that passes a band of frequencies and attenuates frequencies both higher and lower than those in the band that is passed.
- *Cutoff* frequencies are the frequencies defining the boundaries between frequencies that *passed* and *rejected*.

3.9.2 Frequency-Selective Filters

The frequency response of a continuous-time **ideal** lowpass filter is

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}, \qquad (3.140)$$

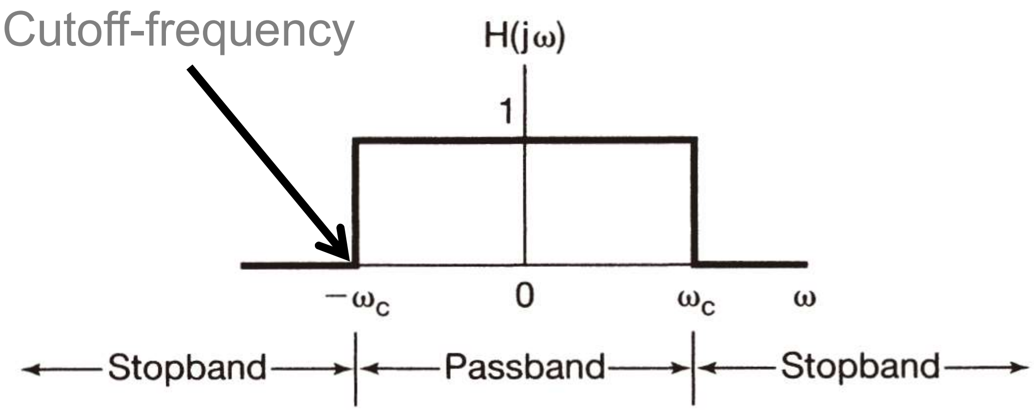


Figure 3.26 Frequency response of an ideal lowpass filter.

3.9.2 Frequency-Selective Filters

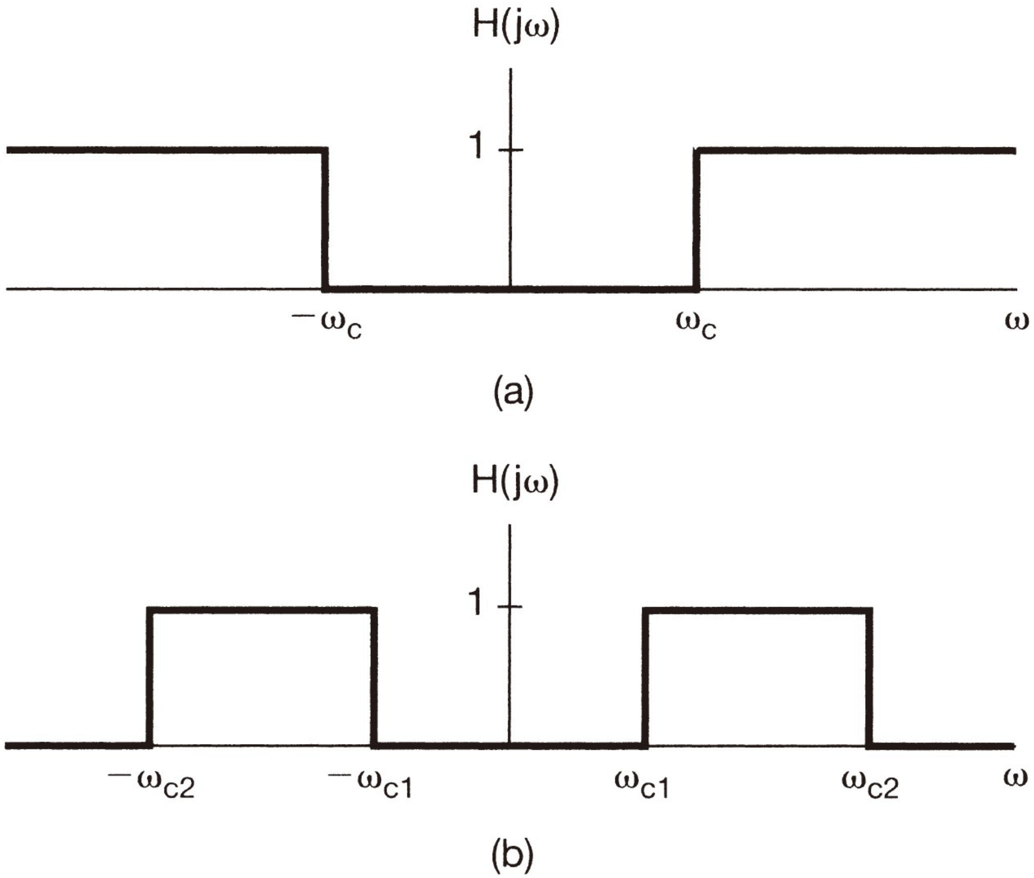


Figure 3.27 (a) Frequency response of an ideal highpass filter; (b) frequency response of an ideal bandpass filter.

3.9.2 Frequency-Selective Filters

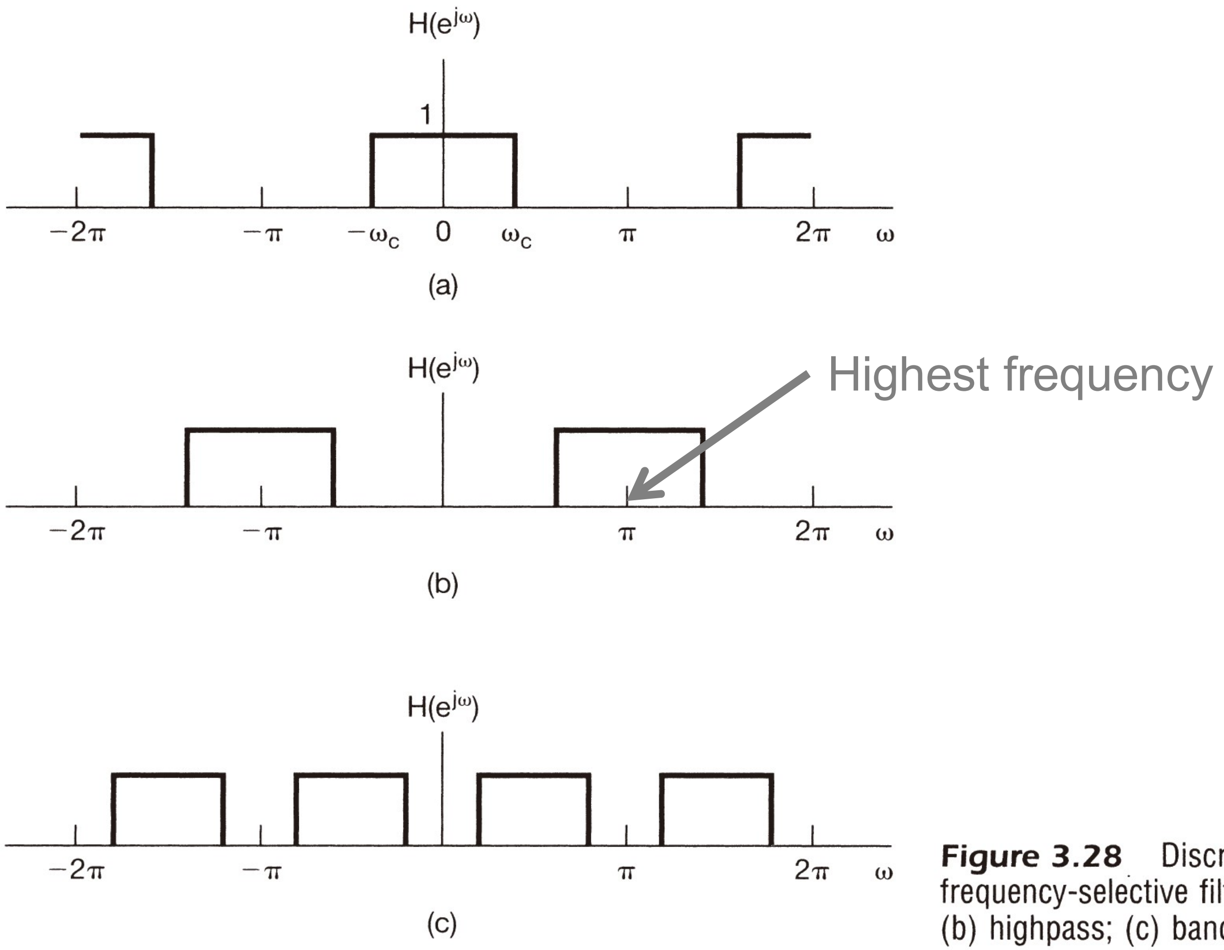


Figure 3.28 Discrete-time ideal frequency-selective filters: (a) lowpass; (b) highpass; (c) bandpass.

3.9.2 Frequency-Selective Filters

Idea filter are quite useful in describing idealized system configurations for variety of applications. However, in practice, they should be **approximated** in order to be realized.

-You will learn a bit more about this in the future, and other course such as communication.

3.10 Examples of Continuous-Time Filters Described by Differentail Equations

In many applications, frequency-selective filtering is accomplished through the use of LTI systems described by **linear constant-coefficient differential or difference** equations.

- Many physical systems are characterized by these equations (suspension system in Ch.6)
- These equations can be implemented using either analog or digital hardware.
- There equations are flexible to describe a large range of filters (e.g., they can well approximate ideal lowpass filter)

3.10.1 A Simple RC Lowpass Filter

In this case, the output voltage is related to the input voltage through the linear constant-coefficient differential equation

$$RC\frac{dv_c(t)}{dt}+v_c(t)=v_s(t). \quad (3.141)$$

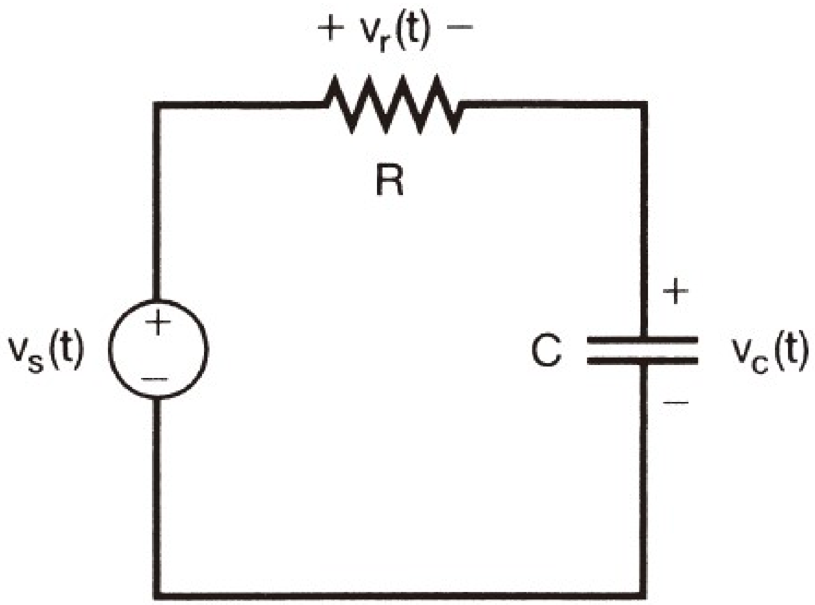


Figure 3.29 First-order *RC* filter.

3.10.1 A Simple RC Lowpass Filter

In order to determine its frequency response *H(jω)*, we note that, by definition, with input voltage $v_s(t) = e^{j\omega t}$, we must have the output voltage .

$$v_c(t) = H(j\omega)e^{j\omega t} \tag{3.142}$$

or

$$RC\frac{d}{dt}\left[H(j\omega)e^{j\omega t}\right] + H(j\omega)e^{j\omega t} = e^{j\omega t},$$

$$RCj\omega H(j\omega)e^{j\omega t} + H(j\omega)e^{j\omega t} = e^{j\omega t}, \tag{3.143}$$

3.10.1 A Simple RC Lowpass Filter (not ideal)

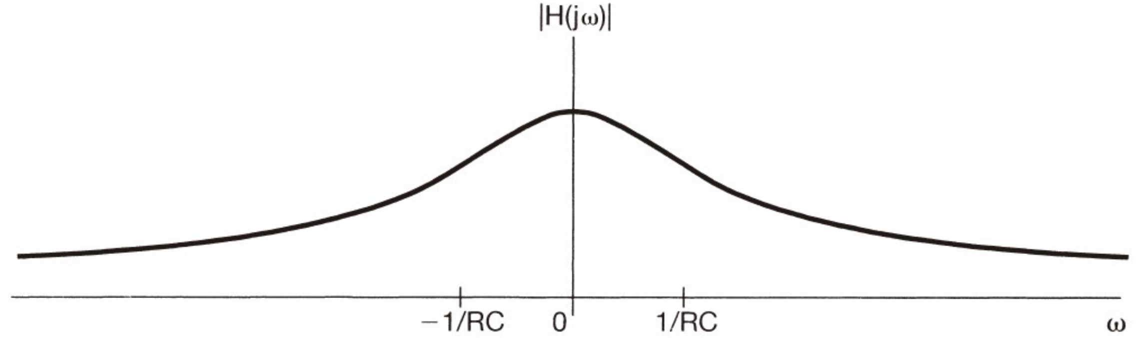
From which it follows directly that

$$H(j\omega)e^{j\omega t} = \frac{1}{1+RCj\omega}e^{j\omega t}, \qquad (3.144)$$

or

$$H(j\omega) = \frac{1}{1+RCj\omega}. \qquad (3.145)$$

- $\omega = 0$, $|H(j\omega)| \approx 1$
- When *ω is large,* $|H(j\omega)|$ *becomes smaller*



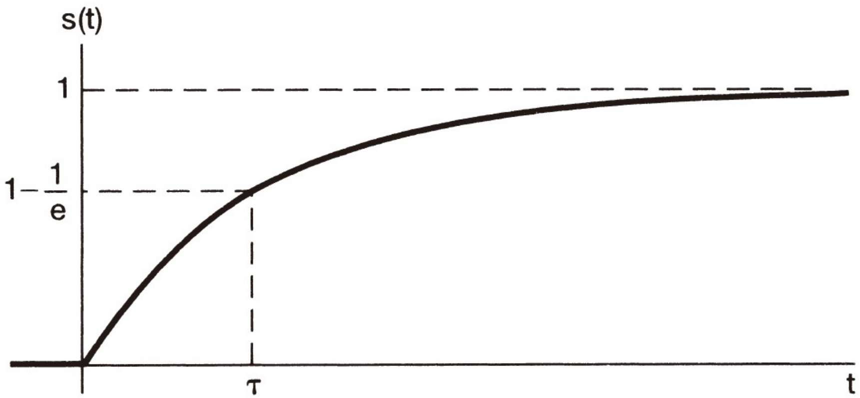
3.10.1 A Simple RC Lowpass Filter

To provide a first glimpse at the trade-offs involved in filter design, let us briefly consider the **time-domain** behavior of the circuit. In particular, the impulse response of the system described by eq. (3.141) is

$$h(t) = \frac{1}{RC} e^{-t/RC} u(t), \qquad (3.146)$$

and the step response is

$$s(t) = \left[1 - e^{-t/RC}\right] u(t), \qquad (3.147)$$

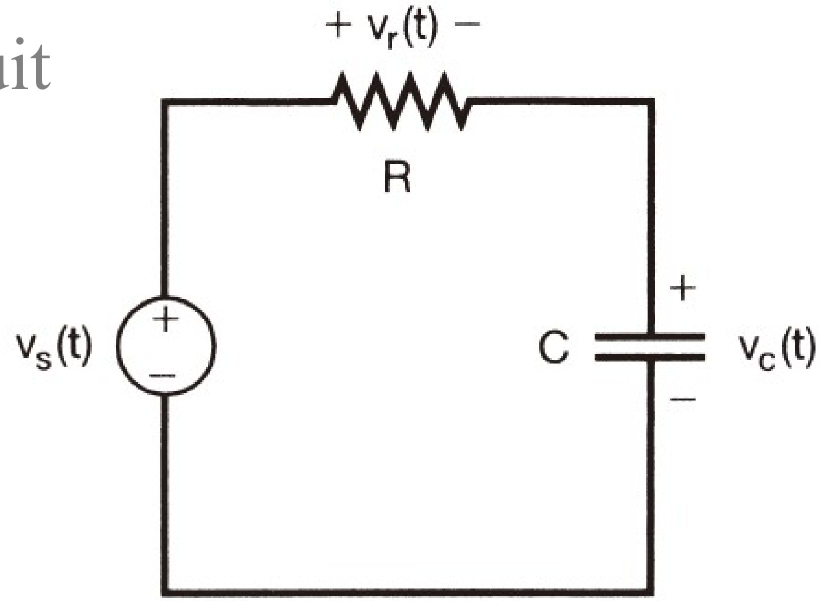


where $\tau = RC$

3.10.2 A Simple RC Highpass Filter

We can use the same RC circuit but use the voltage across the resistor $v_r(t)$ as output.

In this case, the differential equation relating input and output is



$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}.$$

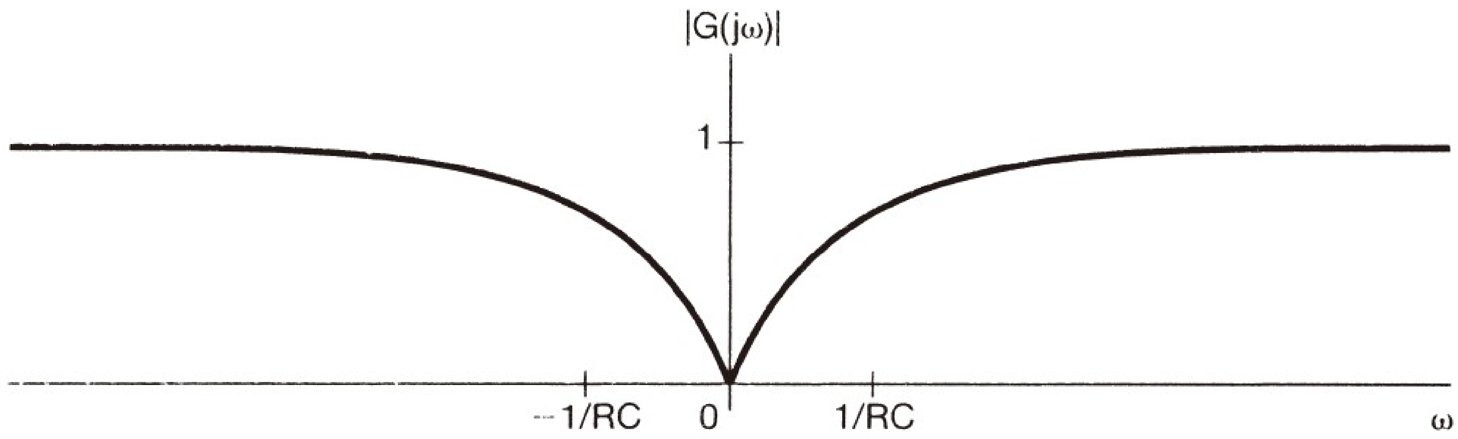
(3.148)

3.10.2 A Simple RC Highpass Filter (not ideal)

Find the frequency response $G(j\omega)$ of this system in exactly the same way we did in the previous case:

If $v_s(t) = e^{j\omega t}$, then we must have $v_r(t) = G(j\omega)e^{j\omega t}$

$$G(j\omega) = \frac{j\omega RC}{1 + j\omega RC}. \tag{3.149}$$



3.10.2 A Simple RC Highpass Filter

$$V_c(t) = \left[1 - e^{-t/RC}\right]u(t),$$

From Figure 3.29, we see that $v_r(t) = v_s(t) - v_c(t)$. Thus, if $v_s(t) = u(t), v_c(t)$ must be given by eq. (3.147).

$$v_r(t) = e^{-t/RC}u(t), \qquad (3.150)$$

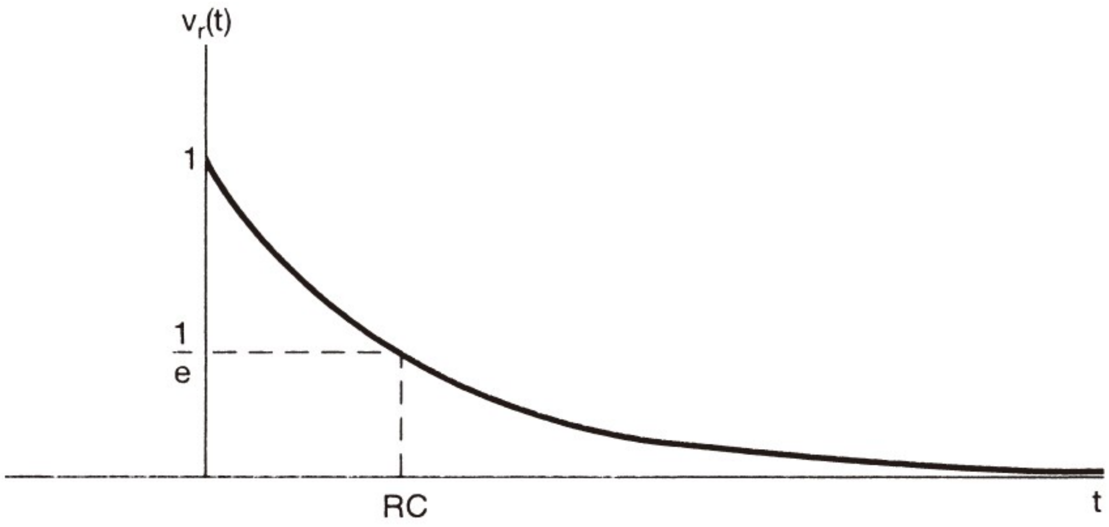


Figure 3.33 Step response of the first-order *RC* highpass filter with $\tau = RC$.

3.11 Examples of Discrete-Time Filters Described by Difference Equations

Discrete-time linear constant coefficient difference equations can represent two types of filters:

- **IIR** system: recursive and have Infinite-length Impulse Response
- **FIR** system: nonrecursive and have finite-length Impulse Response

3.11.1 First-Order Recursive Discrete-Time Filters

$$y[n] - ay[n-1] = x[n].$$

if $x[n] = e^{j\omega n}$, then $y[n] = H(e^{j\omega})e^{j\omega n}$, where $H(e^{j\omega})$ is the frequency response of the system.

$$H(e^{j\omega})e^{j\omega n} - aH(e^{j\omega})e^{j\omega(n-1)} = e^{j\omega n}, \qquad (3.152)$$

or

$$\left[1 - ae^{-j\omega}\right]H(e^{j\omega})e^{j\omega n} = e^{j\omega n}, \qquad (3.153)$$

3.11.1 First-Order Recursive Discrete-Time Filters

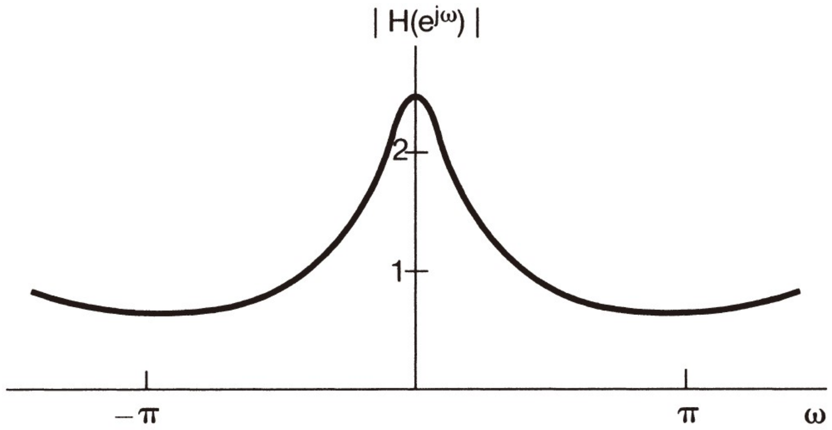
$$\left[1 - ae^{-j\omega}\right] H(e^{j\omega}) e^{j\omega n} = e^{j\omega n},$$

So that

$$H(e^{j\omega}) = \frac{1}{1 - ae^{-j\omega}}. \qquad (3.154)$$

The frequency response of system in Eq. (3.151)

When $0 < a < 1$, it is an approx. lowpass filter.



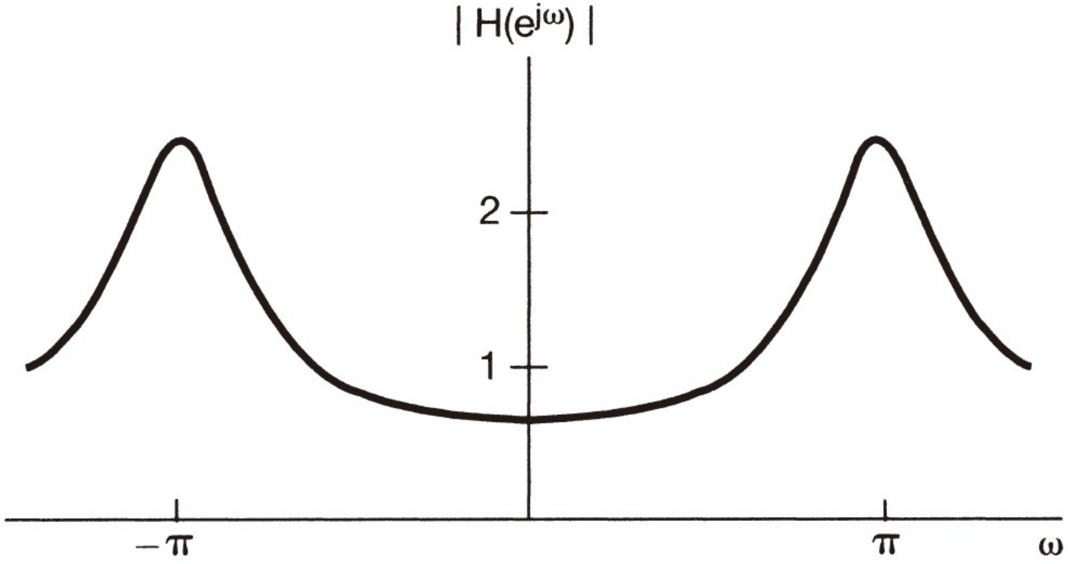
$$H(e^{j0}) = \frac{1}{1 - ae^{-j0}} = \frac{1}{1 - a}.$$

$$H(e^{j\pi}) = \frac{1}{1 - ae^{-j\pi}} = \frac{1}{1 + a}.$$

$$a = 0.6$$

3.11.2 Nonrecursive Discrete-Time Filters

When $-1 < a < 0$, it is an approx. highpass filter.



$$H(e^{j0}) = \frac{1}{1 - ae^{-j0}} = \frac{1}{1-a}.$$

$$H(e^{j\pi}) = \frac{1}{1 - ae^{-j\pi}} = \frac{1}{1+a}.$$

$$a = -0.6$$

3.11.2 Nonrecursive Discrete-Time Filters

The general form of an FIR nonrecursive difference equation is

$$y[n] = \sum_{k=-N}^{M} b_k x[n-k]. \tag{3.157}$$

An only slightly more complex example is the three-point moving-average filter, which is of the form

$$y[n] = \frac{1}{3}\left(x[n-1] + x[n] + x[n+1]\right), \tag{3.158}$$

3.11.2 Nonrecursive Discrete-Time Filters

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h[n]e^{-j\omega n} \quad (3.122)$$

so that each output y[n] is the average of three consecutive input values. In this case, impulse response (finite-length)

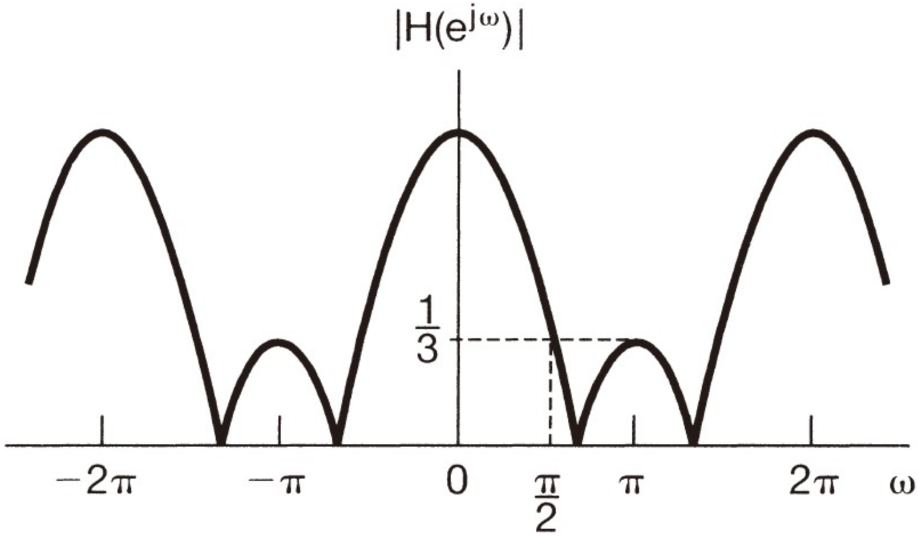
$$h[n] = \frac{1}{3}\left[\delta[n+1] + \delta[n] + \delta[n-1]\right],$$

and thus, from eq. (3.122), the corresponding frequency response is

$$H(e^{j\omega}) = \frac{1}{3}\left[e^{j\omega} + 1 + e^{-j\omega}\right] = \frac{1}{3}(1 + 2\cos\omega). \qquad (3.159)$$

3.11.2 Nonrecursive Discrete-Time Filters

$$H(e^{j\omega}) = \frac{1}{3}\left[e^{j\omega} + 1 + e^{-j\omega}\right] = \frac{1}{3}(1 + 2\cos\omega).$$



$$H(e^{j0}) = \frac{1}{3}\left[e^{j0} + 1 + e^{-j0}\right] = 1.$$

$$H(e^{j\pi}) = \frac{1}{3}\left[e^{j\pi} + 1 + e^{-j\pi}\right] = \frac{1}{3}.$$

Figure 3.35 Magnitude of the frequency response of a three-point moving-average lowpass filter.

3.11.2 Nonrecursive Discrete-Time Filters

$$H(e^{j\omega}) = \frac{1}{3}\left[e^{j\omega} + 1 + e^{-j\omega}\right] = \frac{1}{3}\left(1 + 2\cos\omega\right).$$

From (3.159), we can see the filter has no parameter to adjust the cutoff frequency

To overcome this, consider averaging over N + M + 1 neighboring points—that is, using a difference equation of the form

$$y[n] = \frac{1}{N+M+1}\sum_{k=-N}^{M} x[n-k]. \qquad (3.160)$$

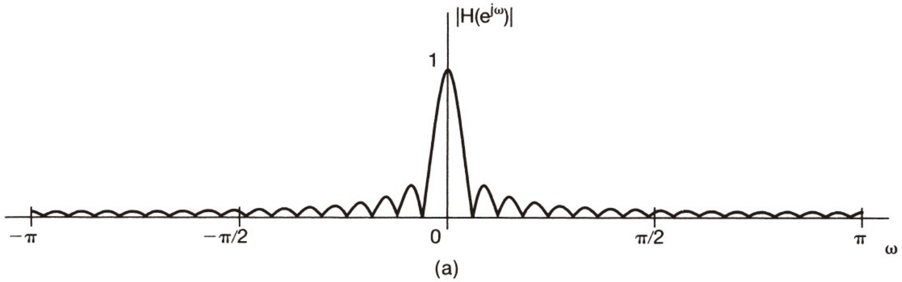
The filter's frequency response is

$$H\left(e^{j\omega}\right) = \frac{1}{N+M+1}\sum_{k=-N}^{M} e^{-j\omega k}. \qquad (3.161)$$

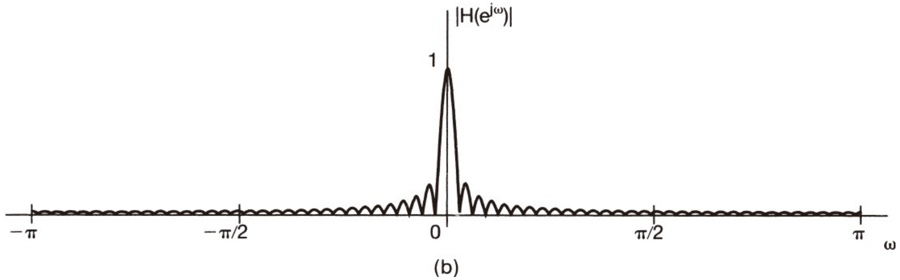
3.11.2 Nonrecursive Discrete-Time Filters

The summation in eq.(3.161) can be evaluated by performing calculations similar to those in Example 3.12, yielding

$$H(e^{j\omega}) = \frac{1}{N+M+1} e^{j\omega[(N-M)/2]} \frac{\sin[\omega(M+N+1)/2]}{\sin(\omega/2)}. \tag{3.162}$$



M=N=16



M=N=32

3.11.2 Nonrecursive Discrete-Time Filters

Nonrecursive filters can also be used to perform highpass filtering operations. To illustrate this, again with a simple example, consider the difference equation

$$y[n] = \frac{x[n] - x[n-1]}{2}. \qquad (3.163)$$

In this case, $h[n] = \frac{1}{2}\{\delta[n] - \delta[n-1]\}$ (finite-length) , so that direct application of eq. (3.122) yields

$$H(e^{j\omega}) = \frac{1}{2}\left[1 - e^{-j\omega}\right] = je^{j\omega/2}\sin(\omega/2). \qquad (3.164)$$

3.11.2 Nonrecursive Discrete-Time Filters

$$H(e^{j\omega}) = \frac{1}{2}\left[1 - e^{-j\omega}\right] = je^{j\omega/2}\sin(\omega/2).$$

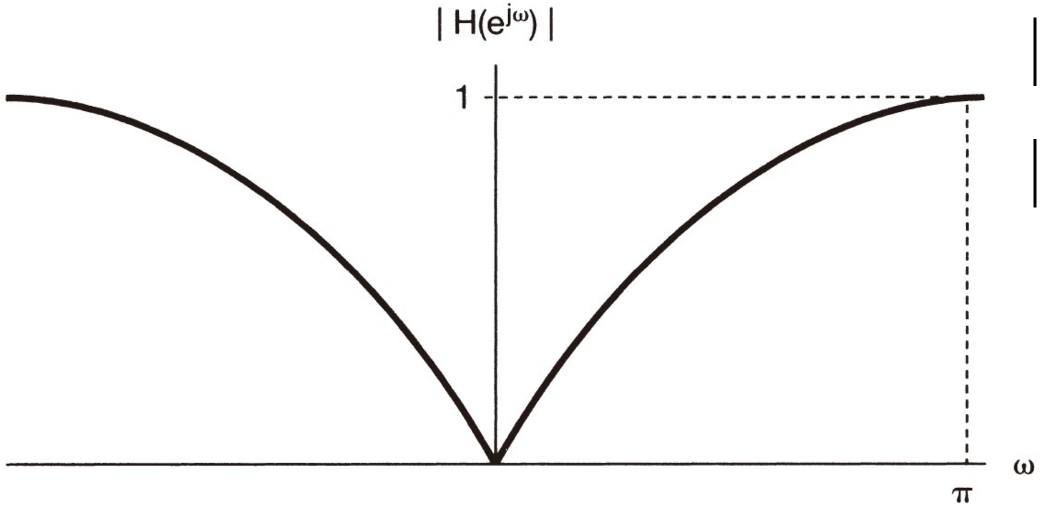


Figure 3.37 Frequency response of a simple highpass filter.

$$|H(e^{j0})| = |e^{j0/2}|\sin(0/2) = 0.$$

$$|H(e^{j\pi})| = |e^{j\pi/2}|\sin(\pi/2) = 1.$$

3.12 Summary

- History of **Fourier Series** (FS)
- Motivation of using FS: complex exponential are **eignfunctions** of LTI system $e^{st} \rightarrow y(t) = H(s)e^{st}$
- Any periodic signal of practical usage can be represented by FS (**convergence**)
- How to obtain **Fourier coefficient**
- Properties of FS: linearity, time-shifting, etc.
- **Frequency response** of a LTI system
- **Filtering** of signals using LTI system: frequency-shaping, frequency-selective