Chapter 2
Linear Time-Invariant Systems

Min Sun

2.1.1 The Representation of Discrete-Time Signals in Terms of **Impulses**

Sample/Sifting property

$$x[-1]\delta[n+1]=\begin{cases}x[-1], & n=-1\\ 0, & n\neq -1\end{cases},$$

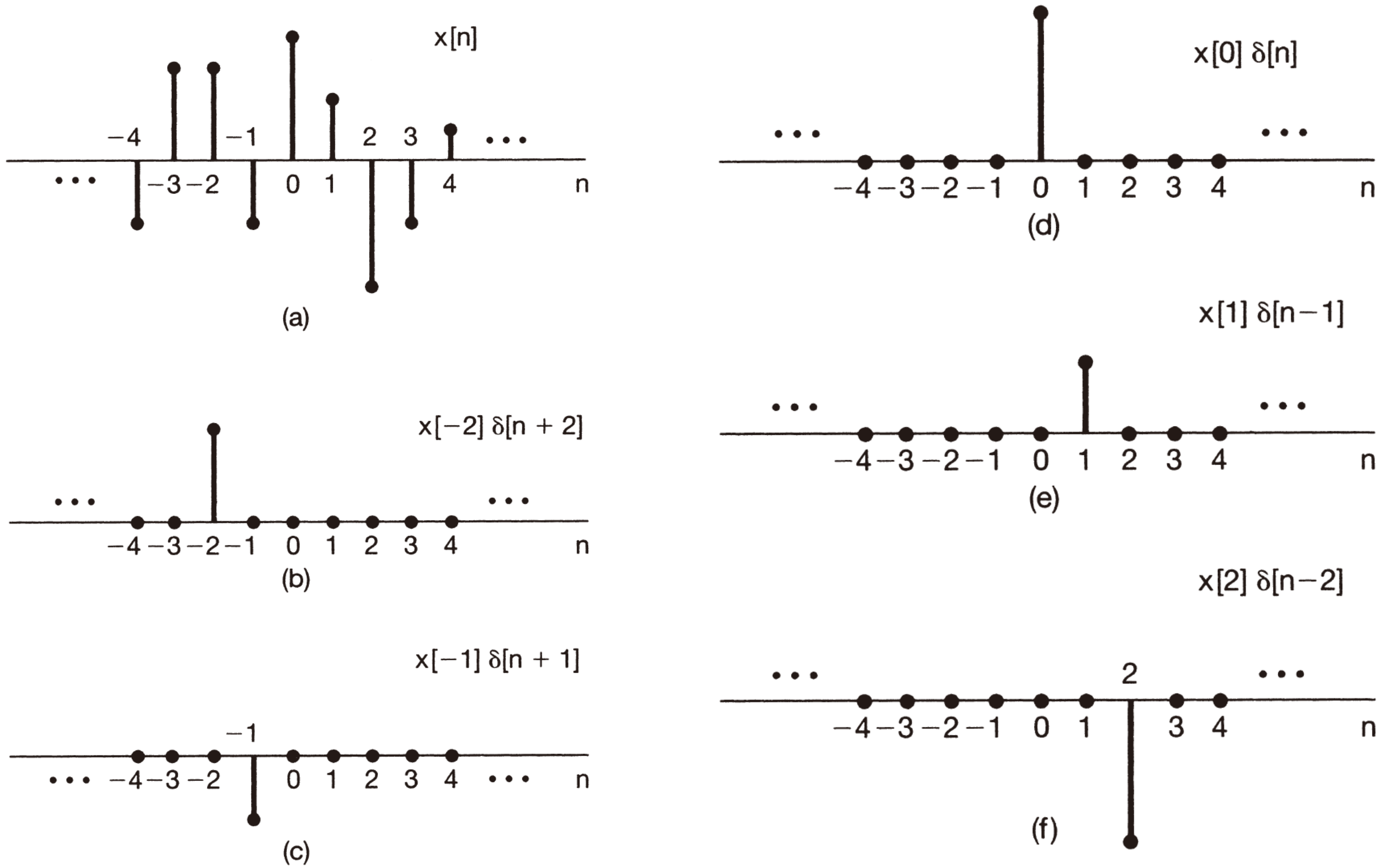
$$x[0]\delta[n]=\begin{cases}x[0], & n=0\\ 0, & n\neq 0\end{cases},$$

$$x[1]\delta[n-1]=\begin{cases}x[1], & n=1\\ 0, & n\neq 1\end{cases}.$$

All signal x[n] can be represented by sum of impulses with different magnitude.

將任何訊號$x[n]$以一連串不同大小的脈衝函數的合成來表示。

2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses



2.1.1 The Representation of Discrete-Time Signals in Terms of Impulses

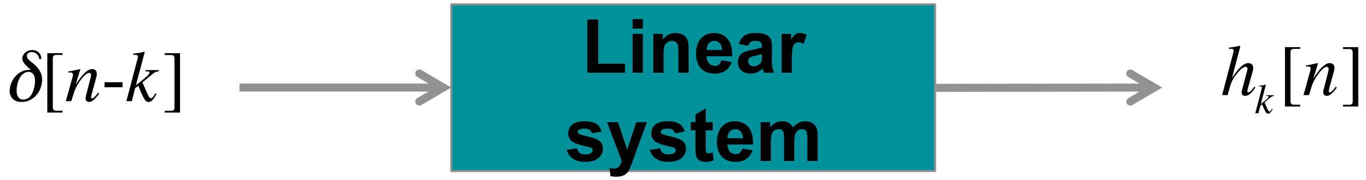
- **Sifting property:**

$$x[n] = \ldots + x[-3]\delta[n+3] + x[-2]\delta[n+2] + x[-1]\delta[n+1] + x[0]\delta[n] + x[1]\delta[n-1] + x[2]\delta[n-2] + x[3]\delta[n-3] + \ldots \quad (2.1)$$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]. \quad (2.2)$$

任何序列$x[n]$均可用不同時間移位的單位脈衝$\delta[n-k]$的線性組合來表示。

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems



Impulse response $h_k[n]$ denotes the response of a linear system to the impulse $\delta[n-k]$

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]. \qquad (2.2)$$

Since every x[n] equals to the sum of all scaled impulse $\delta[n-k]$, using the linearity of the system, the response of x[n] is the sum of all scaled $h_k[n]$ as follows,

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]. \qquad (2.3)$$

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

- **Time invariant** system:

  Since $\delta[n-k]$ is a time-shifted version of $\delta[n-0]=\delta[n]$, the response $h_k[n]$ is a time-shifted version of $h_0[n]$

$$h_k[n] = h_0[n-k] \qquad (2.4)$$

若系統為非時變則$h_k[n]$將與$\delta[n]$為輸入在時間n-k處的響應相同，即：$h_k[n] = h_0[n-k]$

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

we will drop the subscript on $h_0[n]$ and define the unit impulse response

$$h[n] = h_0[n] \qquad (2.5)$$

將$h_0[n]$簡寫成$h[n]$

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n]. \qquad (2.3)$$

2.1.2 The Discrete-Time Unit Impulse Response and the Convolution-Sum Representation of LTI Systems

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h_k[n] \quad (2.3)$$

$$h_k[n] = h_0[n-k] = h[n-k] \quad (2.4)$$

- **Convolution**

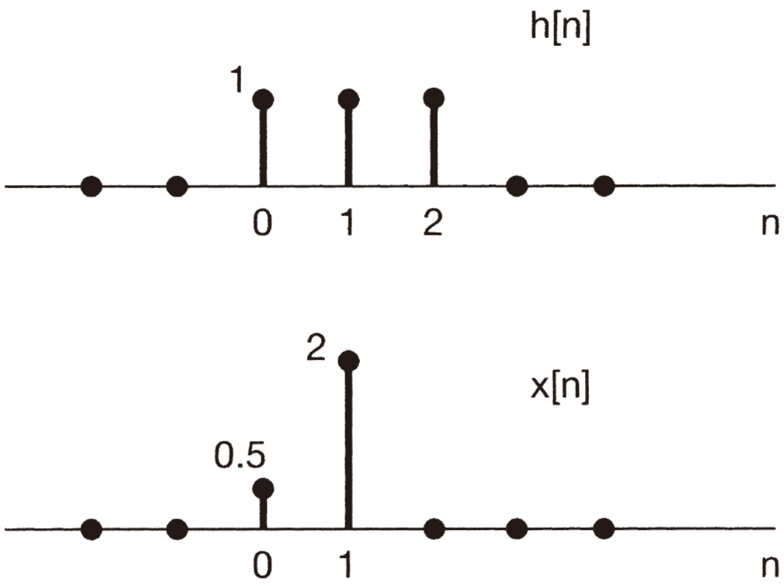
$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k]. \qquad (2.6)$$

$$y[n] = x[n] * h[n]. \qquad (2.7)$$

LTI 系統的輸出與任意輸入及單位脈衝響應的關係式

Example 2.1

- Consider an LTI system with impulse response $h[n]$ and input $x[n]$, as illustrated in Figure 2.3(a). For this case, since only $x[0]$ and $x[1]$ are nonzero, eq.(2.6) simplifies to the expression



(a)

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1] \qquad (2.8)$$

Example 2.1

$$y[n] = x[0]h[n-0] + x[1]h[n-1]$$
$$= 0.5h[n] + 2h[n-1]$$

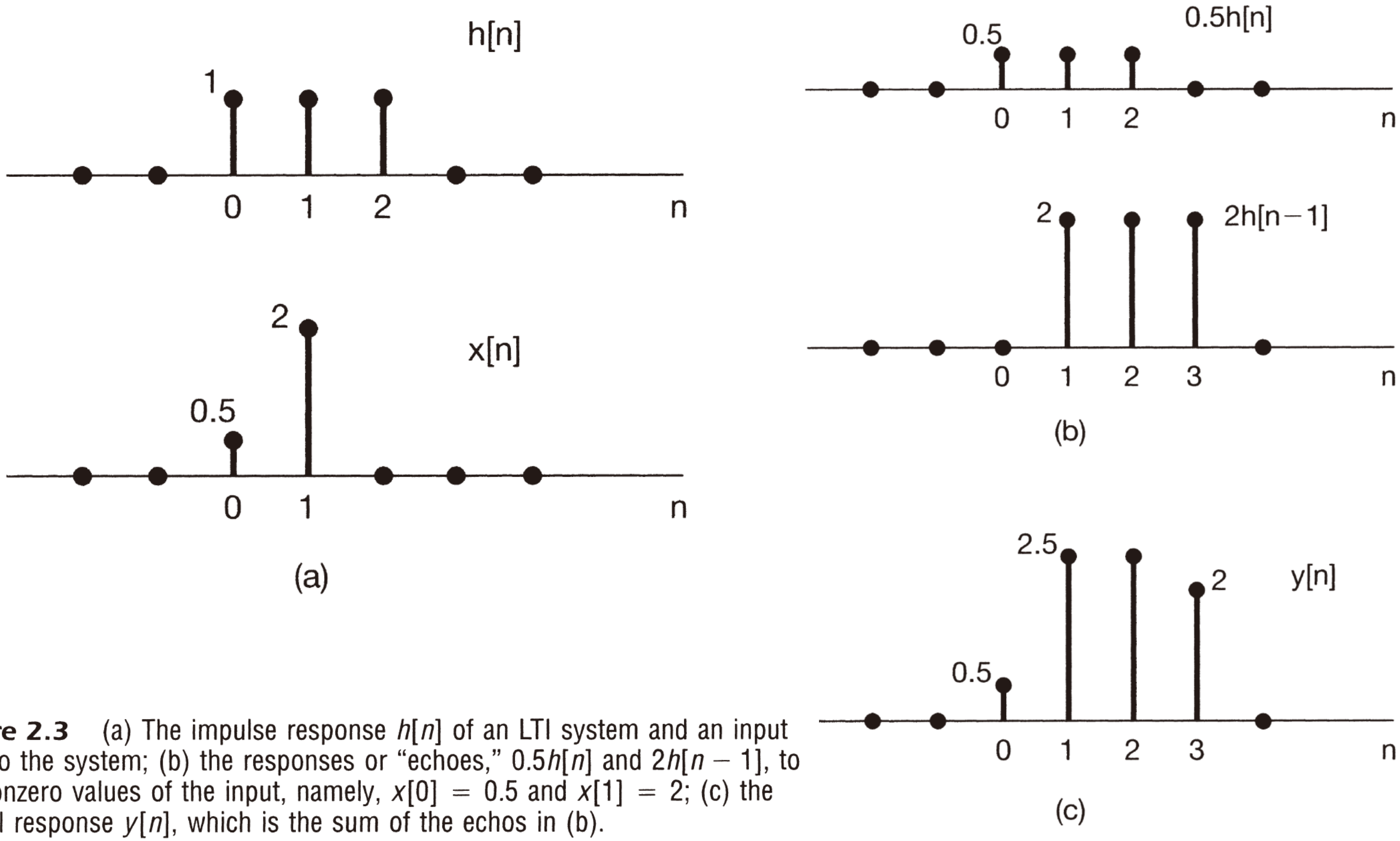
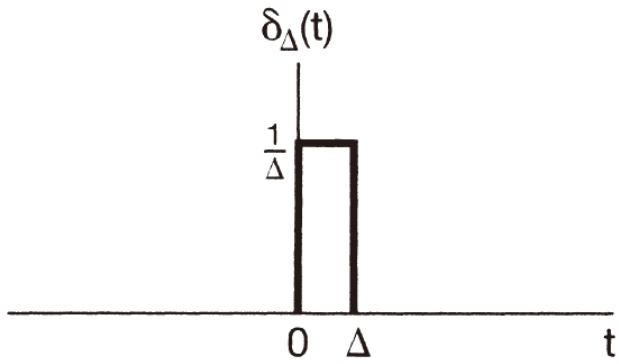


Figure 2.3 (a) The impulse response $h[n]$ of an LTI system and an input $x[n]$ to the system; (b) the responses or "echoes," $0.5h[n]$ and $2h[n-1]$, to the nonzero values of the input, namely, $x[0] = 0.5$ and $x[1] = 2$; (c) the overall response $y[n]$, which is the sum of the echos in (b).

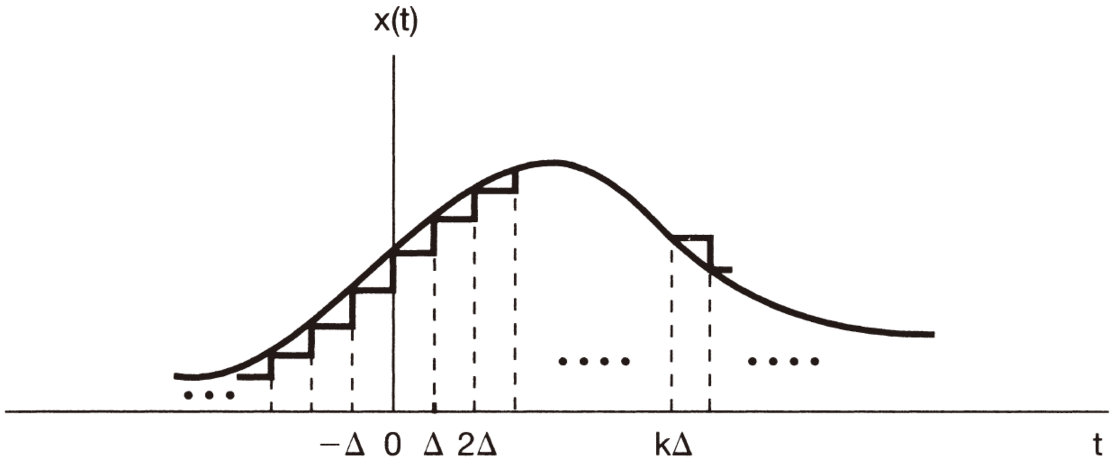
2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \le t < \Delta \\ 0, & \text{otherwise} \end{cases} \qquad (2.24)$$



then, since $\Delta\delta_\Delta(t)$ has **unit amplitude**, we have the expression

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\Delta. \qquad (2.25)$$



2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

- **Sifting property:**

$$x[t] = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta. \qquad (2.26)$$

$$x(t) = \int_{-\infty}^{+\infty} x(\tau)\delta(t-\tau)\ d\tau. \qquad (2.27)$$

訊號*x(t)*的脈衝函數表示法

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta[n-k]. \qquad (2.2)$$

2.2.1 The Representation of Continuous-Time Signals in Terms of Impulses

We note that, for the specific example of $x(t) = u(t)$, eq. (2.27) becomes

$$u(t) = \int_{-\infty}^{+\infty} u(\tau)\delta(t-\tau)d\tau = \int_{0}^{\infty} \delta(t-\tau)d\tau, \qquad (2.28)$$

since $u(\tau) = 0$ for $\tau < 0$ and $u(\tau) = 1$ for $\tau > 0$. Equation (2.28) is identical to eq.(1.75), derived in Section 1.42

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

Let's define $\hat{h}_{k\Delta}(t)$ as the response of a **linear** system to the input $\delta_{\Delta}(t-k\Delta)$. Then from eq.(2.25) and the superposition property, for continuous-time linear systems, we see that

$$\hat{x}(t)=\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta. \qquad (2.25)$$

$$\hat{y}(t) \qquad \hat{h}_{k\Delta}(t)\Delta. \qquad (2.29)$$

$$x(t)=\lim_{\Delta\to 0}\sum_{k=-\infty}^{\infty}x(k\Delta)\delta_{\Delta}(t-k\Delta)\Delta. \qquad (2.25)$$

$$y(t)= \qquad \hat{h}_{k\Delta}(t)\Delta. \qquad (2.30)$$

2.2.2 The Continuous-Time Unit Impulse Response and the Convolution Integral Representation of LTI Systems

$$y(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{+\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta. \qquad (2.30)$$

Therefore, if we let $h_\tau(t)$ denote the response at time t to a unit impulse $\delta(t-\tau)$ located at time τ , then

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h_\tau(t)d\tau. \qquad (2.31)$$

Assuming time invariant and

$$h_\tau(t) = h_0(t-\tau)$$

$$h(t) = h_0(t) \qquad (2.32)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \qquad (2.33)$$

$$y(t) = x(t) * h(t). \quad \text{convolution} \qquad (2.34)$$

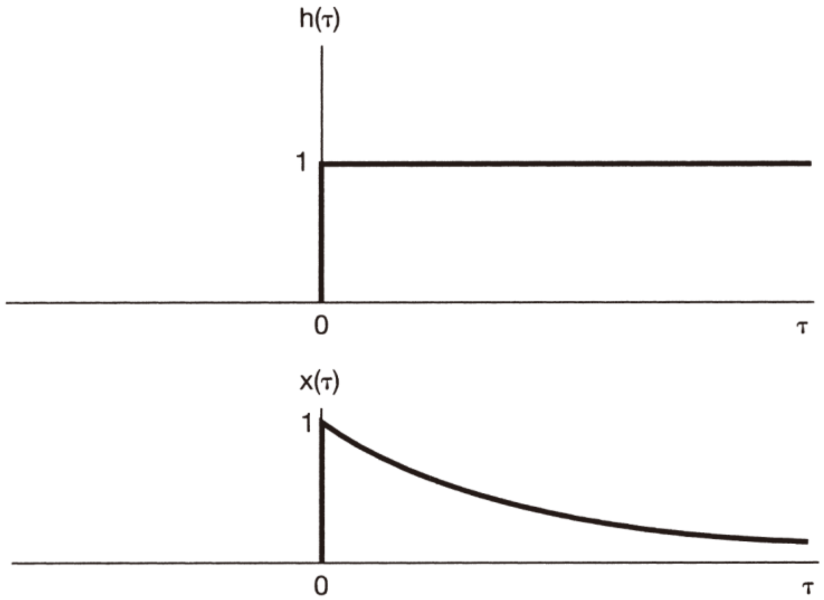
Example 2.6

- Let $x(t)$ be the input to an LTI system with unit impulse response $h(t)$, where

$$x(t) = e^{-at}u(t), \quad a > 0$$

and

$$h(t) = u(t).$$

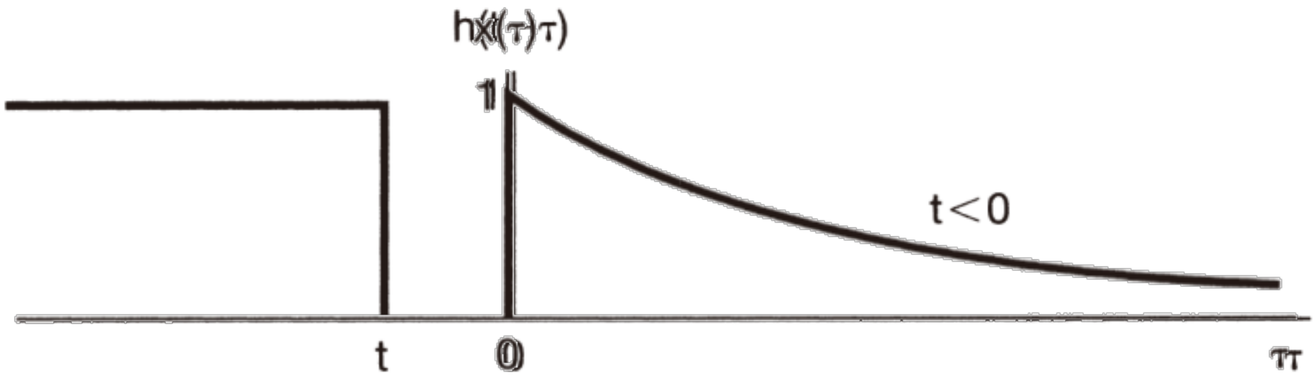


$$x(t) = e^{-at}u(t), \quad h(t) = u(t).$$

Example 2.6

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \qquad (2.33)$$

From this figure, we see that for $t < 0$, the product of $x(\tau)$ and $h(t-\tau)$ is zero, and consequently, $y(t)$ is zero. For $t > 0$.

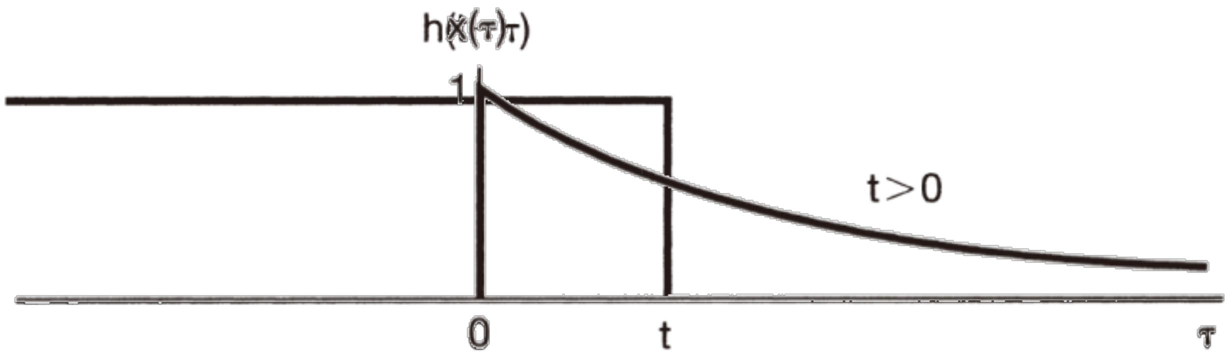


$$x(\tau)h(t-\tau)=0;t<0$$

Example 2.6

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau. \qquad (2.33)$$

For $t > 0$



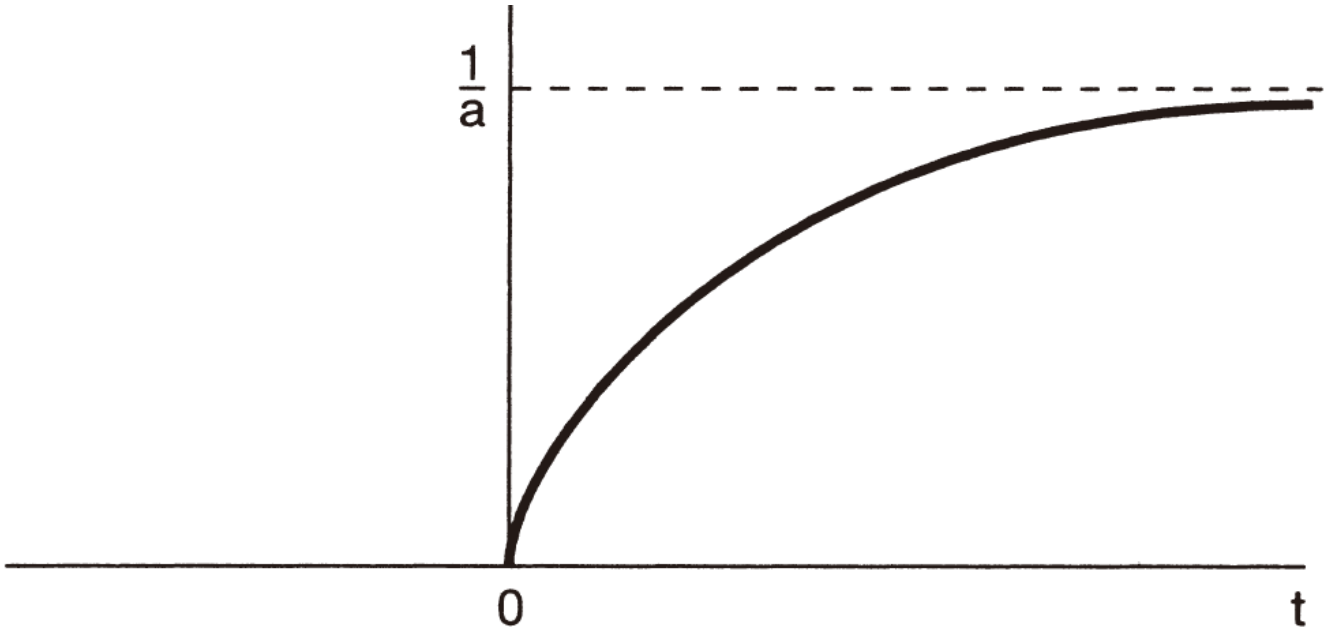
$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}.$$

Example 2.6

For $t > 0$

$$x(\tau)h(t-\tau) = \begin{cases} e^{-a\tau}, & 0 < \tau < t \\ 0, & \text{otherwise} \end{cases}.$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau = \int_{\tau=0}^{t} e^{-a\tau} = -\frac{1}{a}e^{-a\tau}\Big|_0^t = \frac{1}{a}(1-e^{-at})$$



2.3 Properties Of Linear Time-Invariant Systems

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n\text{-}k] = x[n]*h[n] \qquad (2.39)$$

$$y(t) = \int_{-\infty}^{+\infty} x(\tau)h(t\text{-}\tau)d\tau = x(t)*h(t) \qquad (2.40)$$

- The output y of a LTI system is the **convolution sum/integral** of input x with **unit impulse response** h.
- A LTI system can be **completed** determined by its **unit impulse response**.

*This is true only for LTI system

Example 29

- Consider a discrete-time system with unit impulse response

$$h[n] = \begin{cases} 1, & n=0,1 \\ 0, & \text{otherwise} \end{cases}. \qquad (2.41)$$

If the system is LTI, then eq. (2.41) completely determines its input-output behavior.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] + x[n-1]. \quad (2.42)$$

Example 29

- Not for Nonlinear equation

For example, both of the following systems have the same unit impulse response

$$h[n] = \begin{cases} 1, & n=0,1 \\ 0, & \text{otherwise} \end{cases} \qquad (2.41)$$

$$y[n] = (x[n] + x[n-1])^2,$$
$$y[n] = \max(x[n], x[n-1])$$

2.3.1 The Commutative (交換律) Property

$$x[n]*h[n]=h[n]*x[n]=\sum_{k=-\infty}^{+\infty} h[k]x[n\text{-}k], \qquad (2.43)$$

Proof: Set r= n-k, k=n-r

$$x[n]*h[n]=\sum_{k=-\infty}^{+\infty} x[k]h[n\text{-}k]=\sum_{r=n-\infty}^{n+\infty} x[n\text{-}r]h[r]$$

$$=\sum_{r=-\infty}^{+\infty} x[n\text{-}r]h[r]=\sum_{r=-\infty}^{+\infty} h[r]x[n\text{-}r]=h[n]*x[n]$$

Similarly,

$$x(t)*h(t)=h(t)*x(t)=\int_{-\infty}^{+\infty} h(\tau)x(t\text{-}\tau)\,d\tau. \qquad (2.44)$$

2.3.2 The Distributive (分配律) Property

in discrete time

$$x[n]*(h_1[n]+h_2[n]) = x[n]*h_1[n]+x[n]*h_2[n], \quad (2.46)$$

and in continuous time

$$x(t)*[h_1(t)+h_2(t)] = x(t)*h_1(t)+x(t)*h_2(t). \quad (2.47)$$

2.3.2 The Distributive Property

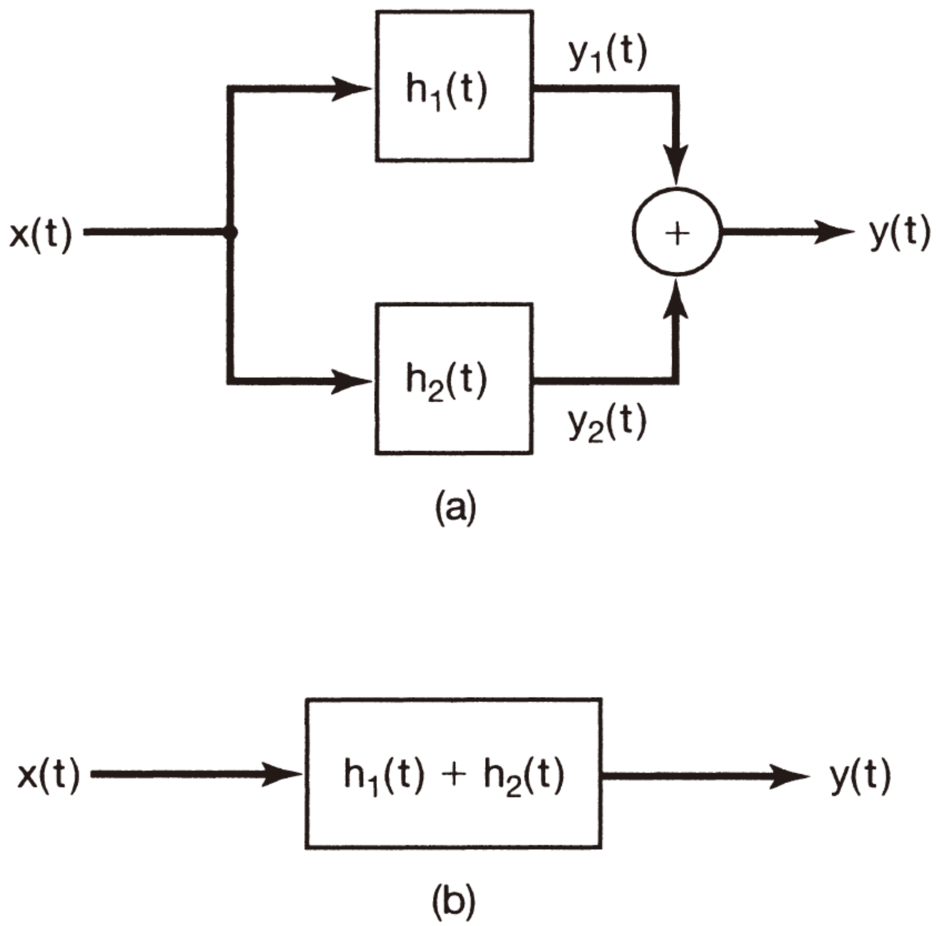


Figure 2.23 Interpretation of the distributive property of convolution for a parallel interconnection of LTI systems.

2.3.2 The Distributive Property

Also, as a consequence of both the commutative and distributive properties, we have

$$[x_1[n] + x_2[n]] * h[n] = x_1[n] * h[n] + x_2[n] * h[n] \qquad (2.50)$$

and

$$[x_1(t) + x_2(t)] * h(t) = x_1(t) * h(t) + x_2(t) * h(t), \qquad (2.51)$$

which simply state that the response of an LTI system to the sum of two inputs must equal the sum of the response to these signals individually.

Example 2.10

Let y[n] denote the convolution of the following two sequences:

$$x[n]=\left(\frac{1}{2}\right)^n u[n]+2^n u[-n] \qquad (2.52)$$

$$h[n]=u[n]. \qquad (2.53)$$

In particular, if we let $x_1[n]=(1/2)^n u[n]$ and $x_2[n]=2^n u[-n]$, it follows that

$$y[n]=(x_1[n]+x_2[n])*h[n]. \qquad (2.54)$$

Example 2.10

$$y[n] = (x_1[n] + x_2[n]) * h[n]. \qquad (2.54)$$

Using the distributive property of convolution, we may rewrite eq.(2.54) as

$$y[n] = y_1[n] + y_2[n] \qquad (2.55)$$

$$y_1[n] = x_1[n] * h[n] \qquad (2.56)$$

$$y_2[n] = x_2[n] * h[n] \qquad (2.57)$$

Eq. 2.56 can be found in example 2.3

Eq. 2.57 can be found in example 2.5

2.3.3 The Association (結合律) Property

in discrete time

$$x[n]*(h_1[n]*h_2[n]) = (x[n]*h_1[n])*h_2[n], \qquad (2.58)$$

and in continuous time

$$x(t)*[h_1(t)*h_2(t)] = [x(t)*h_1(t)]*h_2(t). \qquad (2.59)$$

2.3.3 The Association Property

As a consequence to the associative property, the expressions

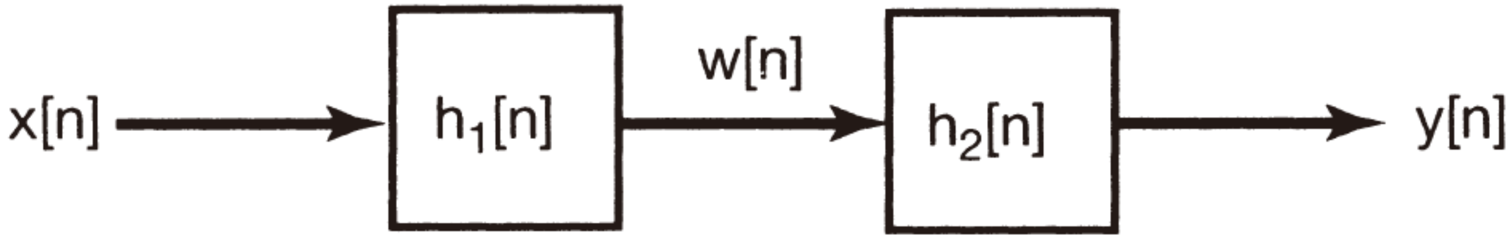
$$y[n] = x[n] * h_1[n] * h_2[n] \qquad (2.60)$$

and

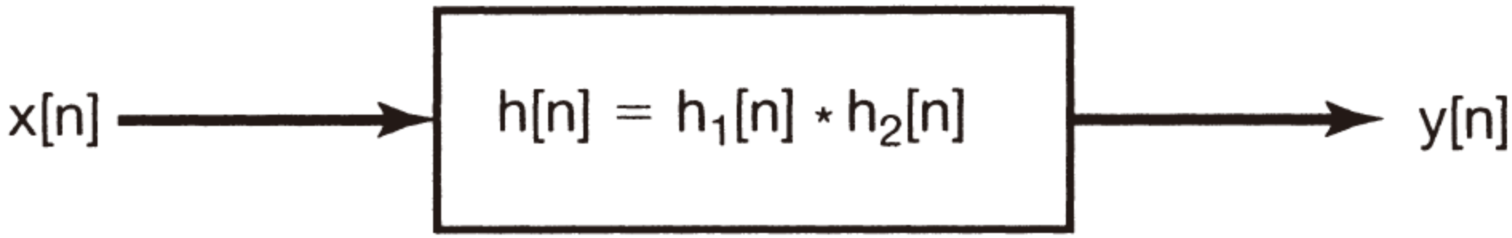
$$y(t) = x(t) * h_1(t) * h_2(t) \qquad (2.61)$$

are unambiguous.

2.3.3 The Association Property

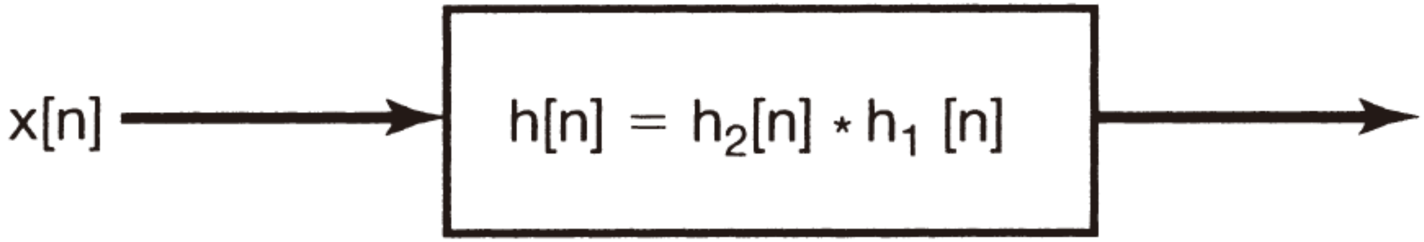


(a)

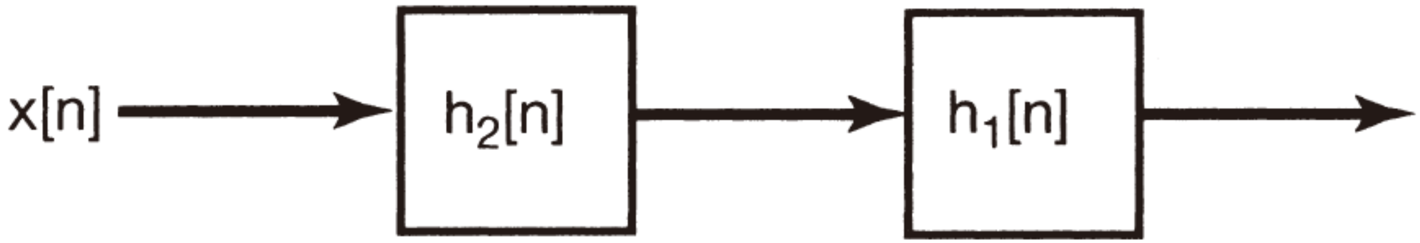


(b)

2.3.3 The Association + Distributive Property



(c)



(d)

2.3.4 LTI Systems with and without Memory

Reall: In chapter 1, we define that a system is memoryless if its output for each value of the independent variable at a given time is dependent on the input at only that same time.

$$y[n] = \sum_{k=-\infty}^{+\infty} x[k]h[n-k] = x[n] * h[n] \qquad (2.39)$$

From Eq. 2.39, a system is memoryless only if $h[n] = 0$ when $n \neq 0$.

2.3.4 LTI Systems with and without Memory

The impulse response has the form

$$h[n] = K\delta[n], \qquad (2.62)$$

where $K = h[0]$ is a constant, and the convolution sum reduces to the relation

$$y[n] = Kx[n]. \qquad (2.63)$$

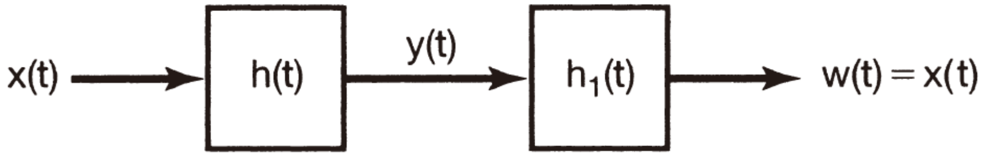
2.3.4 LTI Systems with and without Memory

Similarly, a continuous-time LTI system is memoryless if $h(t) = 0$ for $t \neq 0$, and such a memoryless LTI system has the form

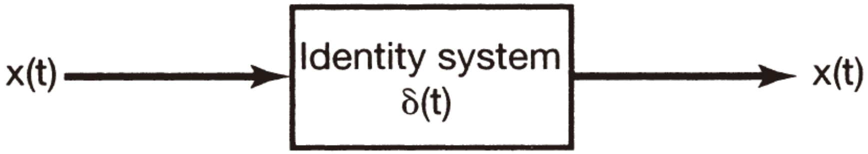
$$y(t) = Kx(t) \qquad (2.64)$$

$$h(t) = K\delta(t) \qquad (2.65)$$

2.3.5 Invertibility of LTI Systems



(a)



(b)

If the impulse response of a LTI system is $h(t)$, the impulse response of its inverse system $h_1(t)$ must satisfy

$$h(t) * h_1(t) = \delta(t). \qquad (2.66)$$

2.3.5 Invertibility of LTI Systems

$$h(t) * h_1(t) = \delta(t). \qquad (2.66)$$

Similarly, the impulse response $h_1[n]$ of the inverse system for an LTI system with impulse response $h[n]$ must satisfy

$$h[n] * h_1[n] = \delta[n]. \qquad (2.67)$$

Example 2.11

- Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0). \qquad (2.68)$$

The impulse response for the system can be obtained from eq.(2.68) by taking the input equal to $\delta(t)$

$$h(t) = \delta(t - t_0). \qquad (2.69)$$

From (2.68) and (2.69)

$$x(t - t_0) = x(t) * \delta(t - t_0). \qquad (2.70)$$

Example 2.11

The convolution of a signal with a shifted impulse simply shifts the signal.

$$x(t-t_0) = x(t) * \delta(t-t_0). \qquad (2.70)$$

$$x(t) * \delta(t-t_0) =$$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-t_0)\,d\tau = x(t-t_0)$$

Example 2.11

- **Inverse system $h_1(t)$:**
  We simply need to shift it back.

  If we take $$h_1(t) = \delta(t + t_0)$$

  then $$h(t) * h_1(t) = \delta(t - t_0) * \delta(t + t_0) = \delta(t)$$

2.3.6 Causality for LTI Systems

- **Recall in chapter 1: A system is causal if the output at any time depends on values of the input at only the present and past times.**

$$y[n] \ \ only \ \ depends \ \ on \ \ x[n-k]; \ \ where \ \ k >= 0$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]. \qquad (2.43)$$

$$implies \ \ that \ \ h[k] = 0 \ \ for \ \ k < 0$$

2.3.6 Causality for LTI Systems

- The impulse response of a causal discrete-time LTI system satisfy the condition

$$h[n] = 0 \quad \text{for } n < 0 \qquad (2.77)$$

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = \sum_{k=-\infty}^{n} x[k]h[n-k], \qquad (2.78)$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k]x[n-k] = \sum_{k=0}^{\infty} h[k]x[n-k]. \qquad (2.79)$$

2.3.6 Causality for LTI Systems

Similarly, a continuous-time LTI system is causal if

$$h(t) = 0 \qquad \text{for } t < 0 \qquad (2.80)$$

同理可得一個因果的連續時間LTI系統必滿足對所有$t < 0$, $h(t)=0$

and the convolution integral is given by

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau. \qquad (2.81)$$

2.3.7 Stability for LTI Systems

- **Recall in chapter 1, a system is stable if every bounded input produces a bounded output.**

$$|x[n]| < B \qquad \text{for all } n \qquad (2.82)$$

- Consider the condition where LTI system is stable.

We obtain an expression for the magnitude of the output:

$$|y[n]| = \left| \sum_{k-\infty}^{+\infty} h[k]x[n-k] \right|. \qquad (2.83)$$

2.3.7 Stability for LTI Systems

$$|x[n]| < B \qquad \text{for all } n \qquad (2.82)$$

- Since the magnitude of the sum of a set of numbers is no larger than the sum of the magnitudes of the numbers

$$|y[n]| \leq \sum_{k=-\infty}^{+\infty} |h[k]||x[n-k]|. \qquad (2.84)$$

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| \qquad for \quad all \quad n \qquad (2.85)$$

2.3.7 Stability for LTI Systems

we can conclude that if the impulse response is **absolutely summable**. That is, if

$$\sum_{k=-\infty}^{+\infty} |h[k]| < \infty \qquad (2.86)$$

The output is bounded.

$$|y[n]| \leq B \sum_{k=-\infty}^{+\infty} |h[k]| < \infty$$

Hence, the system is stable.

(This is also a necessary condition)

2.3.7 Stability for LTI Systems

Similarly, the system is stable if the impulse response is absolutely integrable

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau < \infty \qquad (2.87)$$

Example 2.13

- Consider a system that is a pure time shift in either continuous time or discrete time. Then, in discrete time

$$\sum_{n=-\infty}^{+\infty} |h[n]| = \sum_{-\infty}^{+\infty} |\delta[n-n_0]| = 1 \qquad (2.88)$$

while in continuous time

$$\int_{-\infty}^{+\infty} |h(\tau)| d\tau = \int_{-\infty}^{+\infty} |\delta(\tau - t_0)| d\tau = 1. \qquad (2.89)$$

We conclude both systems are stable.

Example 2.13

An unstable system can also be seen from the fact that its impulse response u[n] is not absolutely summable:

- consider the acumulator

$$y[n]=\sum_{-\infty}^{n} x[n]; \quad x[n]=\delta[n]; \quad h[n]=\sum_{-\infty}^{n} \delta[n]=u[n]$$

$$\sum_{n=-\infty}^{\infty} |u[n]| = \sum_{n=0}^{\infty} u[n] = \infty.$$

- consider the integrator, the continuous-time counterpart of the accumulator:

$$y(t)=\int_{-\infty}^{t} x(\tau)d\tau. \qquad (2.90)$$

Example 2.13

response for the integrator can be found by letting, $x(t) = \delta(t)$ in which case

$$h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$$

and

$$\int_{-\infty}^{+\infty} |u(\tau)| d\tau = \int_{0}^{+\infty} d\tau = \infty$$

Since the impulse response is not absolutely integrable, the system is not stable.

2.3.8 The Unit Step Response of an LTI System

Unit step response s[n] is defined as the system's respond of x[n]=u[n]

- Relation to h[n]

$$y[n] = x[n] * \mathrm{h[n]}; \qquad s[n] = u[n] * h[n]$$

Commutative properties: $s[n] = h[n] * u[n]$

Example 2.13: $$s[n] = \sum_{k=-\infty}^{n} h[k] \qquad (2.91)$$

$$h[n] = s[n] - s[n-1]. \qquad (2.92)$$

2.3.8 The Unit Step Response of an LTI System

Similarly, for continuous signal, we can derive

$$s(t) = \int_{-\infty}^{t} h(\tau) d\tau \qquad (2.93)$$

$$h(t) = \frac{ds(t)}{dt} = s'(t). \qquad (2.94)$$

Conclusions:

•Unit step response can **characterize** an LTI system

2.4.1 Linear Constant-Coefficient Differential Equations

- Systems specified by **linear constant-coefficient differential equations (e.g., Example 1.8)**, let us consider a first-order differential equation as in eq. (1.85), viz.,

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad (2.95)$$

- An **implicit** specification of the system (not a explicit one as y(t)=F(x(t)))

2.4.1 Linear Constant-Coefficient Differential Equations

Insights

- Implicit specification need **auxiliary conditions** to reach explicit specification

E.g., to solve for $V_c(t)$,

We need to solve

$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t) \qquad (1.82)$$

given initial capacitor voltage

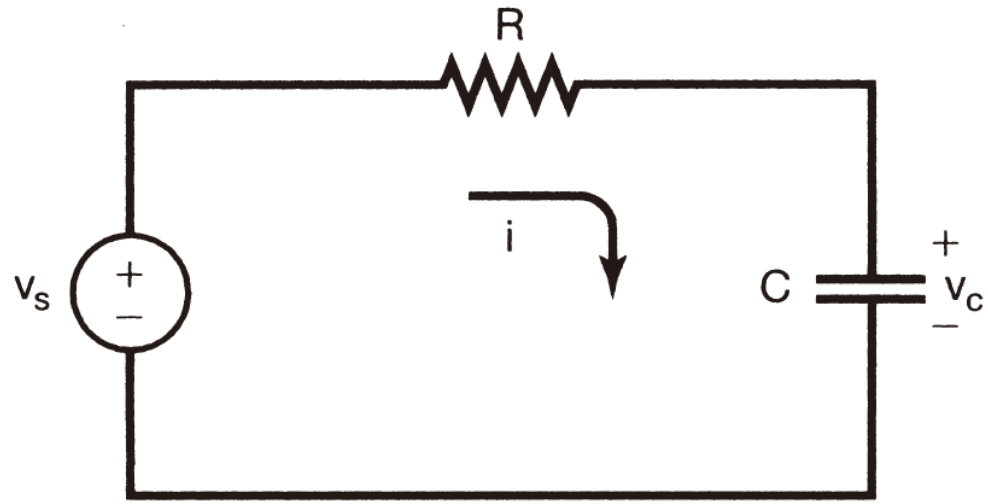


Figure 1.1 A simple *RC* circuit with source voltage v_s and capacitor voltage v_c.

Example 2.14

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad (2.95)$$

Consider the solution of eq.(2.95) when the input signal is

$$x(t) = Ke^{3t}u(t) \qquad (2.96)$$

where K is a real number.

Example 2.14

The complete solution to eq.(2.96) y(t) consists of the sum of a **particular** solution, $y_p(t)$, and a **homogeneous** solution (**natural responses**), $y_h(t)$

$$y(t) = y_p(t) + y_h(t) \qquad (2.97)$$

where $y_p(t)$ is one solution to

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad (2.95)$$

and $y_h(t)$ is the solution to

$$\frac{dy(t)}{dt} + 2y(t) = 0 \qquad (2.98)$$

Example 2.14

$$\frac{dy(t)}{dt} + 2y(t) = x(t) \qquad (2.95)$$

$$x(t) = Ke^{3t}u(t) \qquad (2.96)$$

Given the form of input $x(t) = Ke^{3t}$

we **hypothesize a solution** for $t > 0$ of the form

$$y_p(t) = Ye^{3t} \qquad (2.99)$$

where Y is a number that we must determine.

Substituting eqs: (2.96) and (2.99) into eq.(2.95) for $\boldsymbol{t > 0}$ yields

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t}. \qquad (2.100)$$

Example 2.14

Canceling the factor e^{3t} from both sides of eq.(2.100), we obtain

$$3Y + 2Y = K \qquad (2.101)$$

$$Y = \frac{K}{5} \qquad (2.102)$$

$$y_p(t) = \frac{K}{5} e^{3t}, \qquad t > 0 \qquad (2.103)$$

Example 2.14

$$\frac{dy(t)}{dt} + 2y(t) = 0 \qquad (2.98)$$

In order to determine $y_h(t)$, we hypothesize a solution of the form

$$y_h(t) = Ae^{st} \qquad (2.104)$$

substituting Eq. (2.104) into (2.98)

$$Ase^{st} + 2Ae^{st} = Ae^{st}(s+2) = 0 \qquad (2.105)$$

we get s=-2 and A can be any choice.

we find that the solution of the differential equation for $t > 0$ is

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \qquad t > 0. \qquad (2.106)$$

Example 2.14

- How to solve A? We need an auxiliary condition.
- For causal LTI system, typically we take the condition of initial rest.
- Initial rest: if x(t)=0 for $t<t_0$,then y(t) must equal 0 for $t<t_0$
- For $x(t) = Ke^{3t}u(t)$ $t_0=0$, we need to ensure y(0)=0

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}$$

$$0 = A + \frac{K}{5} \qquad A = -\frac{K}{5} \qquad y(t) = \frac{K}{5}(e^{3t} - e^{-2t})u(t) \qquad (2.108)$$

2.4.1 Linear Constant-Coefficient Differential Equations

- **Higher order differential equations:**

  A general N^{th}-order linear constant-coefficient differential equation is given by

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (2.109)$$

  Note we name it N^{th}-order but not M^{th}-order

Hence below is an example of 0^{th}-order

$$y(t) = \frac{1}{a_0} \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (2.110)$$

2.4.1 Linear Constant-Coefficient Differential Equations

- Solution consists of a **particular** solution and a **homogenous** solution

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0 \qquad (2.111)$$

The solutions to this equation referred to as the **natural responses** of the system.

- **Initial rest condition:**

if x(t)=0 for $t<t_0$,then y(t) must equal 0 for $t<t_0$

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0 \qquad (2.112)$$

2.4.2 Linear Constant-Coefficient Difference Equations

The discrete-time counterpart of eq.(2.109) is the N^{th}-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (2.113)$$

Similarly, solution consists of a **particular** solution and a **homogenous** solution

$$\sum_{k=0}^{N} a_k y[n-k] = 0 \qquad (2.114)$$

Initial rest condition:

if x[n]=0 for $n<=n_0$,then y[n] must equal 0 for $n<=n_0$

2.4.2 Linear Constant-Coefficient Difference Equations

- **Special solution for difference equations**:

  Eq.(2.113) can be rearranged in the form

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\} \qquad (2.115)$$

It is called a **recursive** equation, since it specifies a recursive procedure for determining the output in terms of the input and previous outputs.

$y[n-k]$ for k=1:N auxiliary conditions

2.4.2 Linear Constant-Coefficient Difference Equations

In the special case when $N = 0$, eq.(2.115) reduces to

$$y[n] = \sum_{k=0}^{M} \left(\frac{b_k}{a_0} \right) x[n-k] \qquad (2.116)$$

Eq.(2.116) describes and LTI system, and by direct computation, the impulse response of this system is found to be

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M. \\ 0, & \text{otherwise} \end{cases} \qquad (2.117)$$

No auxiliary conditions are needed.

2.4.2 Linear Constant-Coefficient Difference Equations

$$h[n] = \begin{cases} \frac{b_n}{a_0}, & 0 \le n \le M. \\ 0, & \text{otherwise} \end{cases} \quad (2.117)$$

The impulse response for it has **finite** duration (0<=n<=M); that is, it is nonzero only over a finite time interval. Because of this property, the system specified by eq.(2.116) is often **called a finite impulse response (FIR) system**.

Example 2.15

- Consider the difference equation

$$y[n] - \frac{1}{2} y[n-1] = x[n] \qquad (2.118)$$

Eq.(2.118) can also be expressed in the form

$$y[n] = x[n] + \frac{1}{2} y[n-1] \qquad (2.119)$$

highlighting the fact that we need the previous value of the output, y[n-1], to calculate the current value. **Thus, to begin the recursion, we need an initial condition**.

Example 2.15

suppose that we impose the condition of **initial rest** and consider the input

$$x[n] = K\delta[n] \qquad (2.120)$$

$$y[0] = x[0] + \frac{1}{2}y[-1] = K, \qquad (2.121)$$

$$y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K \qquad (2.122)$$

$$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 K \qquad (2.123)$$

$$y[n] = x[n] + \frac{1}{2}y[n-1] = \left(\frac{1}{2}\right)^n K \qquad (2.124)$$

Example 2.15

Setting $K = 1$, we see that the impulse response for the system considered in this example is

$$h[n] = \left(\frac{1}{2}\right)^n u[n] \qquad (2.125)$$

Such systems are commonly referred to as **infinite impulse response (IIR)** systems

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

Linear constant-coefficient difference/differential equations can be represented by **block diagram**

- A pictorial representation for understanding the behavior and properties of the system
- Valuable for simulation or implementation of the system

Here we only focus on first-order system.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

We begin with the discrete-time case and, in particular, the causal system described by the first-order difference equation

$$y[n] + ay[n-1] = bx[n] \qquad (2.126)$$

rewrite this equation in the form that directly suggests a recursive algorithm for computing successive values of the output $y[n]$

$$y[n] = -ay[n-1] + bx[n] \qquad (2.127)$$

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

$$y[n] = -ay[n-1] + bx[n] \qquad (2.127)$$

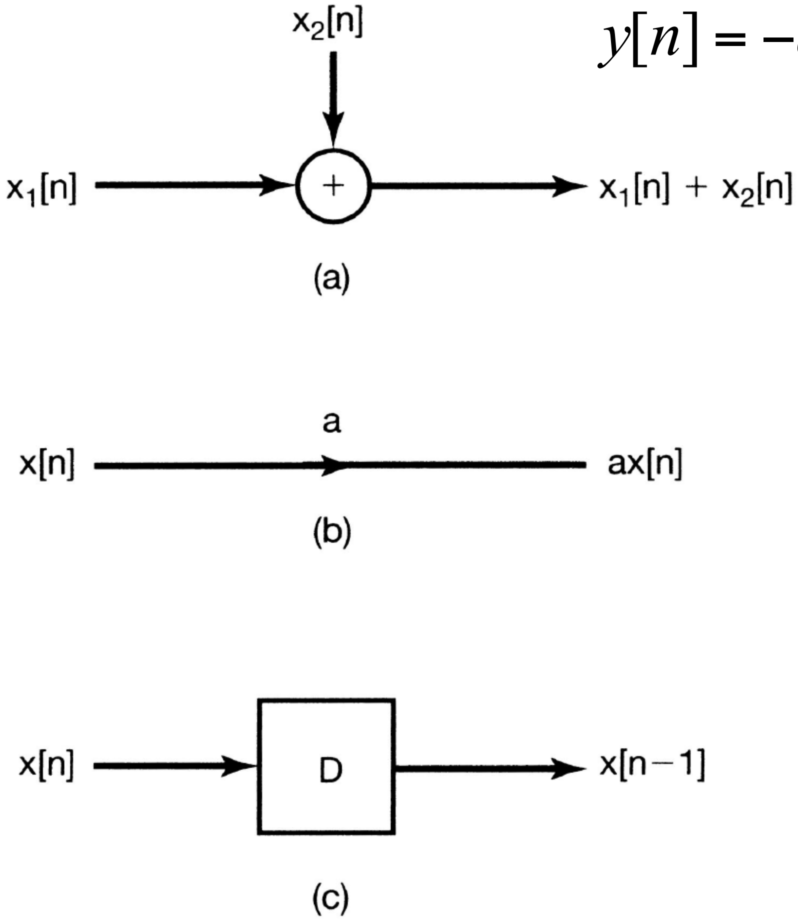
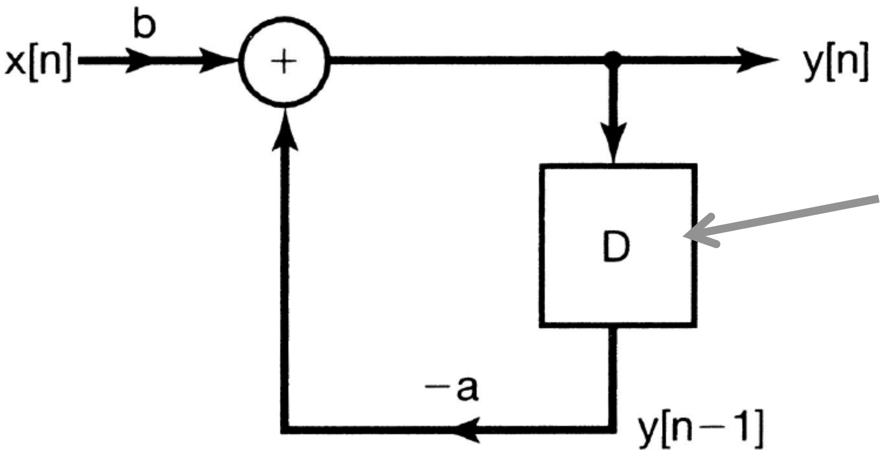


Figure 2.27 Basic elements for the block diagram representation of the causal system described by eq. (2.126): (a) an adder; (b) multiplication by a coefficient; (c) a unit delay.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

$$y[n] = -ay[n-1] + bx[n] \qquad (2.127)$$



Require a memory element and an initial condition

Figure 2.28 Block diagram representation for the causal discrete-time system described by eq. (2.126).

A feedback system

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

Consider next the causal continuous-time system described by a first-order differential equation:

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \qquad (2.128)$$

rewrite it as

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t) \qquad (2.129)$$

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$

訊號合成

訊號乘以係數（放大）

訊號的微分

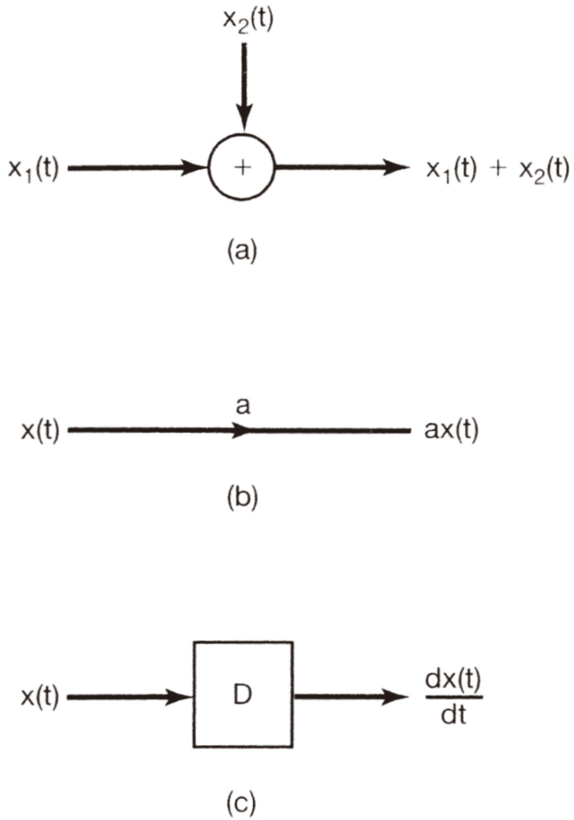
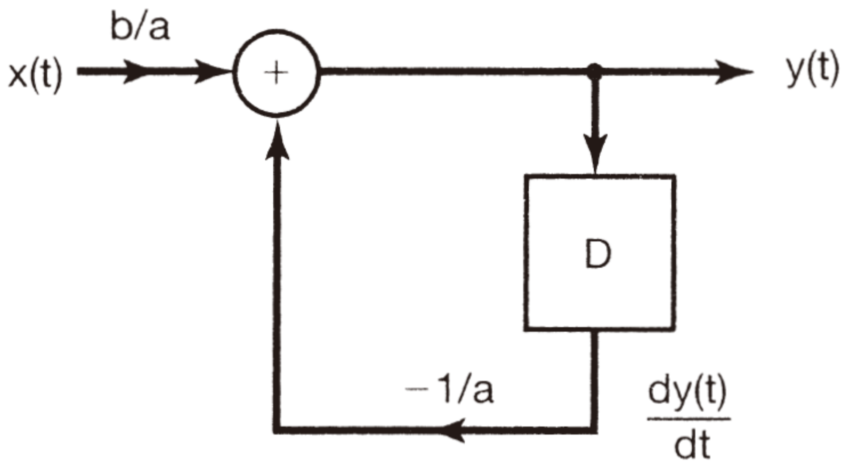


Figure 2.29 One possible set of basic elements for the block diagram representation of the continuous-time system described by eq. (2.128): (a) an adder; (b) multiplication by a coefficient; (c) a differentiator.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations



$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$

Figure 2.30 Block diagram representation for the system in eqs. (2.128) and (2.129), using adders, multiplications by coefficients, and differentiators.

However, differentiator is difficult to implement and sensitive to noise and errors.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

$$\frac{dy(t)}{dt} + ay(t) = bx(t) \qquad (2.128)$$

An alternative way

$$\frac{dy(t)}{dt} = bx(t) - ay(t) \qquad (2.130)$$

Consequently, we obtain the equation

$$y(t) = \int_{-\infty}^{t} \left[bx(\tau) - ay(\tau)\right] d\tau \qquad (2.131)$$

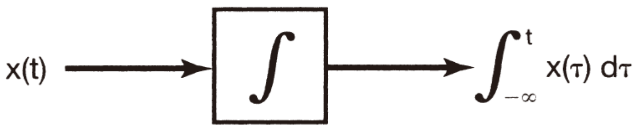


Figure 2.31 Pictorial representation of an integrator.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

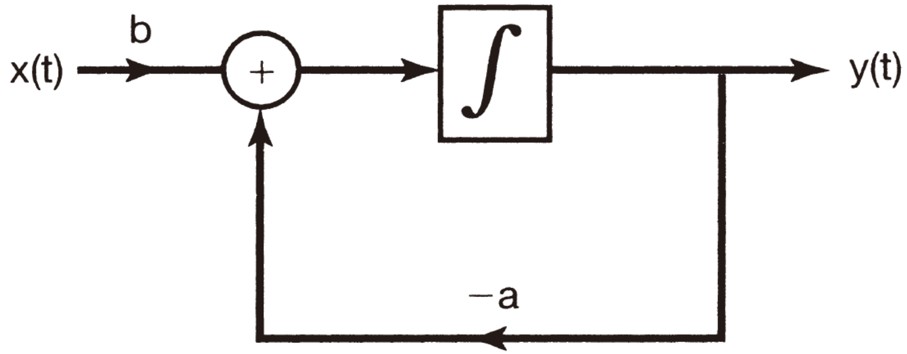


Figure 2.32 Block diagram representation for the system in eqs. (2.128) and (2.131), using adders, multiplications by coefficients, and integrators.

An integrator can be implemented using an operational amplifier.

2.4.3 Block Diagram Representation of First-Order Systems Described by Differential and Difference Equations

$$y(t) = \int_{-\infty}^{t} \left[bx(\tau) - ay(\tau) \right] d\tau \qquad (2.131)$$

This is perhaps more readily seen if we consider integrating eq.(2.130) from a finite point in time t_0, resulting in the expression

$$y(t) = y(t_0) + \int_{t_0}^{t} \left[bx(\tau) - ay(\tau) \right] d\tau \qquad (2.132)$$

We need initial condition (i.e., $y(t_0)$) to solve for $y(t)$

2.6 Summary

- Impulse representation for both discrete and continuous signals
- Unit impulse/step response for LTI system
- Use convolution sum/integral for output of LTI system
- Properties of convolution
- Properties of LTI: memory, causality, etc.
- Linear Constant-Coefficient Differential/ Difference Equations: particular/homogenous solution, auxiliary condition, block diagram, etc.