EE 361002 Signal and System HW6 Answer

3.58. (a) We have

$$z[n+N] = \sum_{<L>} x[r]y[n+N-r].$$

Since $y[n]$ is periodic with period N, $y[n+N-r] = y[n-r]$. Therefore,

$$z[n+N] = \sum_{<L>} x[r]y[n-r] = z[n].$$

Therefore, $z[n]$ is also periodic with period N.

(b) The FS coefficients of $z[n]$ are

$$\begin{aligned} c_l &= \frac{1}{N}\sum_{n=<N>}\sum_{k=<N>} a_k b_{n-k} e^{-j2\pi nl/N} \\ &= \frac{1}{N}\sum_{k=<N>} a_k e^{-j2\pi kl/N} \sum_{n=<N>} b_{n-k} e^{-j2\pi(n-k)l/N} \\ &= \frac{1}{N} N a_l N b_l \\ &= N a_l b_l. \end{aligned}$$

(c) Here, $n = 8$. The nonzero FS coefficients in the range $0 \le k \le 7$ for $x[n]$ are $a_3 = a_5^* = 1/2j$. Note that for $y[n]$, we need only evaluate b_3 and b_5. We have

$$b_3 = b_5^* = \frac{1}{4(1-e^{-j3\pi/4})}.$$

Therefore, the only nonzero FS coefficients in the range $0 \le k \le 7$ for the periodic convolution of these signals are $c_3 = 8a_3b_3$ and $c_5 = 8a_5b_5$.

(d) Here,

$$x[n] \overset{FS}{\longleftrightarrow} a_k = \frac{1}{16j}\left[\frac{1-e^{j(3\pi/7-\pi k/4)4}}{1-e^{-j(3\pi/7-\pi k/4)}} - \frac{1-e^{j(3\pi/7+\pi k/4)4}}{1-e^{-j(3\pi/7+\pi k/4)}}\right]$$

and

$$y[n] \overset{FS}{\longleftrightarrow} b_k = \frac{1}{8}\left[\frac{1-(1/2)^8}{1-(1/2)e^{-jk\pi/4}}\right].$$

Therefore,

$$z[n] = x[n]y[n] \overset{FS}{\longleftrightarrow} 8a_k b_k.$$

3.47. Considering $x(t)$ to be periodic with period 1, the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. If we now consider $x(t)$ to be periodic with period 3, then the the nonzero FS coefficients of $x(t)$ are $b_3 = b_{-3} = 1/2$.

3.48. (a) The FS coefficients of $x[n-n_0]$ are

$$\begin{aligned}\hat{a}_k &= \frac{1}{N}\sum_{n=0}^{N-1} x[n-n_0]e^{-j2\pi nk/N} \\ &= \frac{1}{N}e^{-j\frac{2\pi n_0 k}{N}}\sum_{n=0}^{N-1} x[n]e^{-j2\pi nk/N} \\ &= e^{-j2\pi k n_0/N}a_k\end{aligned}$$

(b) Using the results of part (a), the FS coefficients of $x[n]-x[n-1]$ are given by

$$\hat{a}_k = a_k - e^{-j2\pi k/n}a_k = [1-e^{-j2\pi k/n}]a_k.$$

(c) Using the results of part (a), the FS coefficients of $x[n]-x[n-N/2]$ are given by

$$\hat{a}_k = a_k[1-e^{-jk\pi}] = \begin{cases} 0, & k \text{ even} \\ 2a_k, & k \text{ odd} \end{cases}.$$

(d) Note that $x[n]+x[n+N/2]$ has a period of $N/2$. The FS coefficients of $x[n]+x[n-N/2]$ are given by

$$\hat{a}_k = \frac{2}{N}\sum_{n=0}^{\frac{N}{2}-1}\left[x[n]+x[n+\frac{N}{2}]\right]e^{-j4\pi nk/N} = 2a_{2k}$$

for $0 \le k \le (N/2-1)$.

(e) The FS coefficients of $x^*[-n]$ are

$$\hat{a}_k = \frac{1}{N}\sum_{n=0}^{N-1} x^*[-n]e^{-j2\pi nk/N} = a_k^*.$$

(f) With N even the FS coefficients of $(-1)^n x[n]$ are

$$\hat{a}_k = \frac{1}{N}\sum_{n=0}^{N-1} x[n]e^{-j(2\pi n/N)(k-\frac{N}{2})} = a_{k-N/2}$$

(g) With N odd, the period of $(-1)^n x[n]$ is $2N$. Therefore, the FS coefficients are

$$\hat{a}_k = \frac{1}{2N}\left[\sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi n}{N}(\frac{k-N}{2})} + \sum_{n=0}^{N-1} x[n]e^{-j\frac{2\pi n}{N}(\frac{k-N}{2})}e^{-j\pi(k-N)}\right].$$

Note that for k odd $\frac{k-N}{2}$ is an integer and $k-N$ is an even integer. Also, for k even, $k-N$ is an odd integer and $e^{-j\pi(k-N)}=-1$. Therefore,

$$\hat{a}_k = \begin{cases} a_{\frac{k-N}{2}}, & k \text{ odd} \\ 0, & k \text{ even} \end{cases}.$$

(h) Here.

$$y[n] = \frac{1}{2}[x[n]+(-1)^n x[n]].$$

For N even,

$$\hat{a}_k = \frac{1}{2}[a_k + a_{k-\frac{N}{2}}].$$

For N odd,

$$\hat{a}(k) = \begin{cases} \frac{1}{2}[a_k + a_{\frac{k-N}{2}}], & k \text{ even} \\ \frac{1}{2}a_k, & k \text{ odd} \end{cases}$$

3.33. We will first evaluate the frequency response of the system. Consider an input $x(t)$ of the form $e^{j\omega t}$. From the discussion in Section 3.9.2 we know that the response to this input will be $y(t) = H(j\omega)e^{j\omega t}$. Therefore, substituing these in the given differential equation, we get

$$H(j\omega)j\omega e^{j\omega t} + 4e^{j\omega t} = e^{j\omega t}.$$

Therefore,

$$H(j\omega) = \frac{1}{j\omega + 4}.$$

From eq. (3.124), we know that

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

when the input is $x(t)$. $x(t)$ has the Fourier series coefficients a_k and fundamental frequency ω_0. Therefore, the Fourier series coefficients of $y(t)$ are $a_k H(jk\omega_0)$.

(a) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_1 = a_{-1} = 1/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_1 = a_1 H(j2\pi) = \frac{1}{2(4 + j2\pi)}, \qquad b_{-1} = a_{-1}H(-j2\pi) = \frac{1}{2(4 - j2\pi)}.$$

(b) Here, $\omega_0 = 2\pi$ and the nonzero FS coefficients of $x(t)$ are $a_2 = a_{-2}^* = 1/2j$ and $a_3 = a_{-3}^* = e^{j\pi/4}/2$. Therefore, the nonzero FS coefficients of $y(t)$ are

$$b_2 = a_2 H(j4\pi) = \frac{1}{2j(4 + j4\pi)}, \qquad b_{-2} = a_{-2}H(-j4\pi) = -\frac{1}{2j(4 - j4\pi)},$$

$$b_3 = a_3 H(j6\pi) = \frac{e^{j\pi/4}}{2(4 + j6\pi)}, \qquad b_{-3} = a_{-3}H(-j6\pi) = -\frac{e^{-j\pi/4}}{2(4 - j6\pi)}.$$

3.34. The frequency response of the system is given by

$$H(j\omega) = \int_{-\infty}^{\infty} e^{-4|t|}e^{-j\omega t}dt = \frac{1}{4 + j\omega} + \frac{1}{4 - j\omega}.$$

(a) Here, $T = 1$ and $\omega_0 = 2\pi$ and $a_k = 1$ for all k. The FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \frac{1}{4 + j2\pi k} + \frac{1}{4 - j2\pi k}.$$

(b) Here, $T = 2$ and $\omega_0 = \pi$ and

$$a_k = \begin{cases} 0, & k \text{ even} \\ 1, & k \text{ odd} \end{cases}.$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 0, & k \text{ even} \\ \frac{1}{4+j\pi k} + \frac{1}{4-j\pi k}, & k \text{ odd} \end{cases}.$$

(c) Here, $T = 1$, $\omega_0 = 2\pi$ and

$$a_k = \begin{cases} 1/2, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k}, & k \text{ odd} \end{cases} .$$

Therefore, the FS coefficients of the output are

$$b_k = a_k H(jk\omega_0) = \begin{cases} 1/4, & k = 0 \\ 0, & k \text{ even}, k \neq 0 \\ \frac{\sin(\pi k/2)}{\pi k} \left[\frac{1}{4+j2\pi k} + \frac{1}{4-j2\pi k}\right], & k \text{ odd} \end{cases}$$