Linear Algebra, EE 10810EECS205004

Final Exam

(Dated: Fall, 2020)

Total scores: 120%

1. (±30%) [**True or False**] Note that: a Right answer for +3%; but a Wrong answer for -3% (答錯倒扣).

    (1) Any singular (non-invertible) $n \times n$ matrix over real numbers has at least one eigenvalue.

    (2) If an $n \times n$ matrix is invertible, then it can be diagonalized.

    (3) There is a 3×6 matrix $\overline{\overline{A}}$ over real numbers such that its row space contains $(1, 0, 1, 1, 0, 0)$ and its nullspace contains $(0, 1, 0, 1, 1, 1)$.

    (4) Let $\overline{\overline{A}}$ and $\overline{\overline{B}}$ be two 5×5 matrices over real numbers such that $\overline{\overline{A}}\,\overline{\overline{B}} = -\overline{\overline{B}}\,\overline{\overline{A}}$. Then, either $\overline{\overline{A}}$ or $\overline{\overline{B}}$ is singular.

    (5) $\{\vec{0}\}$ is a subspace of any vector space, where $\vec{0}$ is the zero vector of any vector space.

    (6) If $\overline{\overline{A}}\,\overline{\overline{B}} = \overline{\overline{O}}$, then either $\overline{\overline{A}} = \overline{\overline{O}}$ or $\overline{\overline{B}} = \overline{\overline{O}}$, where $\overline{\overline{O}}$ is the zero matrix.

    (7) A normal matrix is a symmetric matrix.

    (8) Similar matrices have the same eigenvalues and eigenvectors.

    (9) Given $p \times q$ matrices $\overline{\overline{A}}$ and $\overline{\overline{B}}$, then $\overline{\overline{A}}\,\overline{\overline{B}} \neq \overline{\overline{B}}\,\overline{\overline{A}}$ in general but $det(\overline{\overline{A}}\,\overline{\overline{B}}) = det(\overline{\overline{B}}\,\overline{\overline{A}})$.

    (10) A square matrix is orthogonal if its column vectors are orthogonal.

    [**Answers**]

    (1) **T**;

    (2) **F**; e.g., $\overline{\overline{\mathbf{A}}} = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$

    (3) **F**; i.e., $(1, 0, 1, 1, 0, 0) \cdot (0, 1, 0, 1, 1, 1)^t \neq 0$

    (4) **T**;

    (5) **T**;

    (6) **F**; e.g., $\overline{\overline{\mathbf{A}}} = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix}$, $\overline{\overline{\mathbf{B}}} = \begin{bmatrix} 4 & -6 \\ -2 & 3 \end{bmatrix}$, but $\overline{\overline{\mathbf{A}}}\,\overline{\overline{\mathbf{B}}} = \overline{\overline{\mathbf{O}}}$

    (7) **F**; $\overline{\overline{\mathbf{A}}}\,\overline{\overline{\mathbf{A}}}^* = \overline{\overline{\mathbf{A}}}^*\,\overline{\overline{\mathbf{A}}}$, but $\overline{\overline{\mathbf{A}}} \neq \overline{\overline{\mathbf{A}}}^t$

    (8) **F**; Only eigenvalues are the same, but not the eigenvectors.

    (9) **F**; $A_{p\times q} \cdot B_{p\times q}$ may have the problem.

    (10) **F**; It should be orthonormal, not orthogonal.

2. (10%) [**Adjoint Matrix**]

    Assume $\overline{\overline{\mathbf{A}}}$ is an $n \times n$ matrix.

    (a) (5%) Calculate the product of determinants of $\overline{\overline{\mathbf{A}}}$ and its adjoint matrix , i.e., $\overline{\overline{\mathbf{A}}}^*$,

    $$det(\overline{\overline{\mathbf{A}}}) \cdot det(\overline{\overline{\mathbf{A}}}^*) \tag{1}$$

    (b) (5%) Based on (a), Express the determinant of $\overline{\overline{\mathbf{A}}}^*$, in terms of the determinant of $\overline{\overline{\mathbf{A}}}$ and n.

    [**Answers**]

    (a) $det(\overline{\overline{\mathbf{A}}})\, det(\overline{\overline{\mathbf{A}}}^*) = det(\overline{\overline{\mathbf{A}}})\, \overline{det(\mathbf{A})}$ (+2 分) $= |det(\overline{\overline{\mathbf{A}}})|^2$. (+3 分)

    It is not necessary to apply $det(\overline{\overline{\mathbf{A}}})\, det(\overline{\overline{\mathbf{A}}}^*) = det(\overline{\overline{\mathbf{A}}}\,\overline{\overline{\mathbf{A}}}^*)$ (+2 分).

    (b) $\frac{|det(\overline{\overline{\mathbf{A}}})|^2}{det((\overline{\overline{\mathbf{A}}})}$ (+5 分)

3. (20%) [**Hermitian Matrix**]

(a) (10%)
Given a Hermitian matrix, i.e., $\overline{\overline{\mathbf{H}}} = \overline{\overline{\mathbf{H}}}^*$,

$$\overline{\overline{\mathbf{H}}} = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \tag{2}$$

then Find a unitary matrix $\overline{\overline{\mathbf{U}}}$, such that

$$\overline{\overline{\mathbf{U}}}^* \, \overline{\overline{\mathbf{H}}} \, \overline{\overline{\mathbf{U}}} = \overline{\overline{\mathbf{D}}}, \tag{3}$$

where $\overline{\overline{\mathbf{D}}}$ is a diagonal matrix.

(b) (10%)
Use the result of (a) to find a matrix $\overline{\overline{\mathbf{B}}}$ such that

$$\overline{\overline{\mathbf{H}}} = \overline{\overline{\mathbf{B}}}^* \, \overline{\overline{\mathbf{B}}} \tag{4}$$

[**Answers**]

(a) $\overline{\overline{\mathbf{H}}} = \begin{bmatrix} 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix}$, (+7 分) with the eigenvalues: $4, 0, 2$ (+3 分)

Note that as $\overline{\overline{\mathbf{H}}}$ is Hermitian, the corresponding eigenvectors must be orthonormal. (only +3 分 沒有 orthonormal)

(b) One of 4 possible solutions is $\overline{\overline{\mathbf{B}}} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 1 & -i & 0 \end{bmatrix}$ (+10 分)

4. (15%) [**Spectral Theorem**]

(a) (10%) Find an orthogonal matrix $\overline{\overline{\mathbf{P}}}$ that diagonalizes

$$\overline{\overline{\mathbf{S}}} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \tag{5}$$

(b) (5%) Perform the spectral decomposition for the matrix $\overline{\overline{\mathbf{S}}}$.

[**Answers**]

(a) Eigenvalues and the corresponding eigenvectors:

$$\lambda_1 = 2, \quad \vec{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}; \qquad \text{orthornormal} \quad \vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}; \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}; \tag{6}$$

$$\lambda_2 = 8, \quad \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \qquad \text{orthornormal} \quad \vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}; \tag{7}$$

$$\overline{\overline{\mathbf{P}}} = \begin{bmatrix} \frac{-1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{6}} & \frac{1}{\sqrt{3}} \\ 0 & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} \end{bmatrix}, \qquad \overline{\overline{\mathbf{P}}}^* \, \overline{\overline{\mathbf{S}}} \, \overline{\overline{\mathbf{P}}} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 8 \end{bmatrix} \tag{8}$$

(+7 分) (+3 分)

(b)

$$\overline{\overline{\mathbf{S}}} = 2\vec{u}_1^t\,\vec{u}_1 + 2\vec{u}_2^t\,\vec{u}_2 + 8\vec{u}_3^t\,\vec{u}_3 \tag{9}$$

$$= 2\times\frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + 2\times\frac{1}{6}\begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ -2 & -2 & 4 \end{bmatrix} + 8\times\frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{10}$$

(+5 分)

5. (15%) [**Least Squares Approximation**]

Find the parabola $y = C + D\,x + E\,x^2$ that comes closest (least squares error) to the data points: $(x, y) = (-2, 0)$, $(-1, 0)$, $(0, 1)$, $(1, 0)$, and $(2, 0)$.
[**Answers**]
$y = \frac{34}{70} - \frac{10}{70}\,x^2$

6. (15%) [**SVD**]

Consider the matrix:

$$\overline{\overline{\mathbf{A}}}_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}, \tag{11}$$

Find the corresponding *Singular Value Decomposition*, i.e.,

$$\overline{\overline{\mathbf{A}}}_1 = \overline{\overline{\mathbf{U}}}\,\overline{\overline{\mathbf{\Sigma}}}\,\overline{\overline{\mathbf{V}}}^* \tag{12}$$

[**Answers**]

$$\overline{\overline{\mathbf{A}}}_1 = \begin{bmatrix} \frac{3}{\sqrt{10}} & \frac{-1}{\sqrt{10}} \\ \frac{1}{\sqrt{10}} & \frac{3}{\sqrt{10}} \end{bmatrix} \begin{bmatrix} 6\sqrt{10} & 0 & 0 \\ 0 & 3\sqrt{10} & 0 \end{bmatrix} \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ 2 & -2 & 1 \end{bmatrix}$$

(+3 分) (+9 分, 順序錯誤 -2分) (+3 分)

7. (15%) [**Jordan Canonical Form**]

Given

$$\overline{\overline{\mathbf{A}}}_2 = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}. \tag{13}$$

then, Express $\overline{\overline{\mathbf{A}}}_2$ being similar to the matrix $\overline{\overline{\mathbf{J}}}$ with Jordan form, i.e.,

$$\overline{\overline{\mathbf{J}}} = \overline{\overline{\mathbf{M}}}^{-1}\,\overline{\overline{\mathbf{A}}}_2\,\overline{\overline{\mathbf{M}}} \tag{14}$$

[**Answers**]
Eigenvalues and the corresponding eigenvectors:

$$\lambda_1 = 1, \quad \vec{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \tag{15}$$

$$\lambda_2 = 2, \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}; \qquad \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}; \tag{16}$$

$$\overline{\overline{\mathbf{J}}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \qquad \overline{\overline{\mathbf{M}}} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 1 \\ -1 & -3 & -2 \end{bmatrix} \tag{17}$$

(+4 分) (+2, +2, +7 分)