Linear Algebra, EE 10810EECS205004

Final Exam

(Dated: Fall, 2020)

Total scores: 120%

1. ($\pm$30%) [**True or False**] Note that: a Right answer for +3%; but a Wrong answer for -3% (答錯倒扣).

(1) Any singular (non-invertible) $n \times n$ matrix over real numbers has at least one eigenvalue.

F (2) If an $n \times n$ matrix is invertible, then it can be diagonalized.

(3) There is a 3×6 matrix $\overline{\overline{A}}$ over real numbers such that its row space contains $(1, 0, 1, 1, 0, 0)$ and its nullspace contains $(0, 1, 0, 1, 1, 1)$.

(4) Let $\overline{\overline{A}}$ and $\overline{\overline{B}}$ be two 5×5 matrices over real numbers such that $\overline{\overline{A}}\,\overline{\overline{B}} = -\overline{\overline{B}}\,\overline{\overline{A}}$. Then, either $\overline{\overline{A}}$ or $\overline{\overline{B}}$ is singular.

F (5) $\{\vec{0}\}$ is a subspace of any vector space, where $\vec{0}$ is the zero vector of any vector space.

F (6) If $\overline{\overline{A}}\,\overline{\overline{B}} = \overline{\overline{O}}$, then either $\overline{\overline{A}} = \overline{\overline{O}}$ or $\overline{\overline{B}} = \overline{\overline{O}}$, where $\overline{\overline{O}}$ is the zero matrix. $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

F (7) A normal matrix is a symmetric matrix.

(8) Similar matrices have the same eigenvalues and eigenvectors.

(9) Given $p \times q$ matrices $\overline{\overline{A}}$ and $\overline{\overline{B}}$, then $\overline{\overline{A}}\,\overline{\overline{B}} \neq \overline{\overline{B}}\,\overline{\overline{A}}$ in general but $det(\overline{\overline{A}}\,\overline{\overline{B}}) = det(\overline{\overline{B}}\,\overline{\overline{A}})$.

(10) A square matrix is orthogonal if its column vectors are orthogonal.

2. (10%) [**Adjoint Matrix**]

Assume $\overline{\overline{\mathbf{A}}}$ is an $n \times n$ matrix.

(a) (5%) Calculate the product of determinants of $\overline{\overline{\mathbf{A}}}$ and its adjoint matrix , i.e., $\overline{\overline{\mathbf{A}}}^*$,

$$det(\overline{\overline{\mathbf{A}}}) \cdot det(\overline{\overline{\mathbf{A}}}^*) \qquad = \; | \; | \; | \tag{1}$$

(b) (5%) Based on (a), Express the determinant of $\overline{\overline{\mathbf{A}}}^*$, in terms of the determinant of $\overline{\overline{\mathbf{A}}}$ and n.

3. (20%) [**Hermitian Matrix**]

(a) (10%)
Given a Hermitian matrix, i.e., $\overline{\overline{\mathbf{H}}} = \overline{\overline{\mathbf{H}}}^*$,

$$\overline{\overline{\mathbf{H}}} = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \tag{2}$$

then Find a unitary matrix $\overline{\overline{\mathbf{U}}}$, such that

$$\overline{\overline{\mathbf{U}}}^*\,\overline{\overline{\mathbf{H}}}\,\overline{\overline{\mathbf{U}}} = \overline{\overline{\mathbf{D}}}, \quad \begin{bmatrix} 0 & 0 & 1 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix}\begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}\begin{bmatrix} 0 & \frac{-i}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 1 & 0 & 0 \end{bmatrix} \tag{3}$$

where $\overline{\overline{\mathbf{D}}}$ is a diagonal matrix.

(b) (10%)
Use the result of (a) to find a matrix $\overline{\overline{\mathbf{B}}}$ such that

$$\overline{\overline{\mathbf{H}}} = \overline{\overline{\mathbf{B}}}^*\,\overline{\overline{\mathbf{B}}} \tag{4}$$

$\begin{matrix} 0 & 0 & 4 \\ \sqrt{2}i & \sqrt{2} & 0 \\ 0 & 0 & 0 \end{matrix} \qquad \begin{matrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{matrix}$

$H = UDU^*$

$= U\sqrt{D}U^* U\sqrt{D}^* U^*$

$= B^* B$

$$\begin{bmatrix} 0 & \frac{-i}{\sqrt{2}} & \frac{-i}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\sqrt{2}i & \sqrt{2}i \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & \sqrt{2}i \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} & -\sqrt{2}i \end{bmatrix} = \begin{bmatrix} 1 & -i & 0 \\ i & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

4. (15%) [**Spectral Theorem**]

(a) (10%) Find an orthogonal matrix $\overline{\overline{\mathbf{P}}}$ that diagonalizes

$$\overline{\overline{\mathbf{S}}} = \begin{bmatrix} 4 & 2 & 2 \\ 2 & 4 & 2 \\ 2 & 2 & 4 \end{bmatrix}, \tag{5}$$

(b) 15%) Perform the spectral decomposition for the matrix $\overline{\overline{\mathbf{S}}}$.

5. (15%) [**Least Squares Approximation**]

Find the parabola $y = C + Dx + Ex^2$ that comes closest (least squares error) to the data points: $(x, y) = (-2, 0)$, $(-1, 0)$, $(0, 1)$, $(1, 0)$, and $(2, 0)$.

$$\left[\begin{array}{ccc|ccc} 34 & 0 & 10 & 1 & 0 & 0 \\ 0 & 10 & 0 & 0 & 1 & 0 \\ 10 & 0 & 5 & 0 & 0 & 1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{3}{238} & 0 & -\frac{1}{7} \\ 0 & 1 & 0 & 0 & \frac{1}{10} & 0 \\ 0 & 0 & 1 & -\frac{1}{7} & 0 & \frac{17}{35} \end{array}\right]$$

6. (15%) [**SVD**]

Consider the matrix:

$$\overline{\overline{\mathbf{A}}}_1 = \begin{bmatrix} 4 & 11 & 14 \\ 8 & 7 & -2 \end{bmatrix}, \tag{6}$$

Find the corresponding *Singular Value Decomposition*, i.e.,

$$\overline{\overline{\mathbf{A}}}_1 = \overline{\overline{\mathbf{U}}}\,\overline{\overline{\boldsymbol{\Sigma}}}\,\overline{\overline{\mathbf{V}}}^* \tag{7}$$

7. (15%) [**Jordan Canonical Form**]

Given

$$\overline{\overline{\mathbf{A}}}_2 = \begin{bmatrix} 0 & -1 & -1 \\ -3 & -1 & -2 \\ 7 & 5 & 6 \end{bmatrix}. \tag{8}$$

$$\left[\begin{array}{ccc|ccc} 0 & 1 & -2 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 \\ -1 & -3 & 0 & 0 & 0 & 1 \end{array}\right]$$

then, Express $\overline{\overline{\mathbf{A}}}_2$ being similar to the matrix $\overline{\overline{\mathbf{J}}}$ with Jordan form, i.e.,

$$\overline{\overline{\mathbf{J}}} = \overline{\overline{\mathbf{M}}}^{-1}\,\overline{\overline{\mathbf{A}}}_2\,\overline{\overline{\mathbf{M}}} \tag{9}$$