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# Linear Algebra, EE 10810EECS205004

Second Exam (10:10 AM - 1:00 PM, Friday, December 18th, 2020)
(Dated: Fall, 2020)

Total scores: 120

1. (25%) [**Determinant**]

Let matrix $\overline{\overline{B}}$ be formed by the column vectors $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$, i.e.,

$$\begin{bmatrix} 1&0&0&0\\ 3&1&0&0\\ 2&0&1&0\\ 1&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0\\ 3&1&0&0\\ 2&0&1&0\\ 1&0&0&1 \end{bmatrix}$$

$$\overline{\overline{B}} \equiv \begin{pmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4 \end{pmatrix} = \begin{pmatrix} 1&0&0&0\\ 3&1&0&0\\ 2&0&1&0\\ 1&0&0&1 \end{pmatrix}, \quad = \begin{bmatrix} 1&0&0&0\\ 6&1&0&0\\ 4&0&1&0\\ 2&0&0&1 \end{bmatrix}\begin{bmatrix} 1&0&0&0\\ 3&1&0&0\\ 2&0&1&0\\ 1&0&0&1 \end{bmatrix} \tag{1}$$

$$= \begin{pmatrix} 1&0&0&0\\ 9&1&0&0\\ [illegible]&0&1&0\\ 3&0&0&1 \end{pmatrix}$$

(a) (5%) Calculate $det[2 \cdot \overline{\overline{B}}^{(-2)}]$.

(b) (10%) Calculate $\overline{\overline{\mathbf{B}}}^{(-10)}$.

(c) (10%) What is the minimum distant from $\vec{v}_1$ to the subspace of span $= \{\vec{v}_2, \vec{v}_3, \vec{v}_4\}$, where the distance between two column vectors $\vec{x}$ and $\vec{y}$ is defined as $\sqrt{(\vec{x}-\vec{y})^t(\vec{x}-\vec{y})}$, or $||\vec{x}-\vec{y}||$.

2. (10%) [**Distinct Eigenvalues**]

Let $\overline{\overline{\mathbf{A}}}$ be a positive definite $(\forall \lambda_i > 0)$, symmetric matrix $(\overline{\overline{\mathbf{A}}} = \overline{\overline{\mathbf{A}}}^t)$. Prove that eigenvectors corresponding to distinct eigenvalues are orthogonal.

$\lambda_\alpha v \cdot w = \overline{\overline{A}} v \cdot w = (\overline{\overline{A}}^T v^T) w = (vA)^T w = Aw$

$v \cdot \lambda_\beta w$

$\overline{\overline{A}} w \cdot v = A$

$v A w$

$(\overline{\overline{A}} w^T) v = (\lambda_\beta w) \cdot v$

$(wA)^T$

3. (15%) [**Vandermode Matrix**]

$$\overline{\overline{V}} \equiv \begin{vmatrix} 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \\ 1 & x_4 & x_4^2 & x_4^3 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & x_2-x_1 & x_2^2-x_1^2 & x_2^3-x_1^3 \\ 1 & x_3-x_1 & x_3^2-x_1^2 & x_3^3-x_1^3 \\ 1 & x_4-x_1 & x_4^2-x_1^2 & x_4^3-x_1^3 \end{vmatrix} \tag{2}$$

(a) (5%) Find the determinant of Vandermode matrix, i.e., $det[\overline{\overline{V}}]$.

(b) (10%) Find the inverse matrix by using Cramer's rule, i.e., $(\overline{\overline{V}})^{-1} = \overline{\overline{C}}^t / det[\overline{\overline{V}}]$.

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x_2-x_1 & 0 \\ 1 & x_3-x_1 & (x_3^2-x_1^2)-(x_2+x_1) \\ 1 & x_4-x_1 & -(x_4^2-x_1^2) \end{vmatrix}$$

$-(x_2+x_1)(x_3-x_1)$

$x_2 x_3^2 x_4^3 + x_3 x_4^2 x_2^3 + x_4 x_2^2 x_3^3 - x_4 x_3^2 x_2^3 - x_3 x_2^2 x_4^3 - x_2 x_4^2 x_3^3$

$-$

④
$x_2^3 x_1 x_3 (x_3 - x_1)$
$x_3^3 x_1 x_2 (x_1 - x_2)$
$x_1^3 x_2 x_3 (x_2 - x_3)$

③
$x_4^3 x_1 x_2 (x_2 - x_1)$
$x_1^3 x_2 x_4 (x_4 - x_2)$
$x_2^3 x_1 x_4 (x_1 - x_4)$

①
$x_2^3 x_3 x_4 (x_4 - x_3)$
$x_4^3 x_2 x_3 (x_3 - x_2)$
$x_3^3 x_2 x_4 (x_2 - x_4)$

②
$x_3^3 x_1 x_4 (x_4 - x_1)$
$x_1^3 x_3 x_4 (x_3 - x_4)$
$x_4^3 x_1 x_3 (x_1 - x_3)$

$-x_2 x_3 + x_2 x_1 - x_1 x_3 + x_1^2$

$x_3^2 - x_2 x_3 + x_2 x_1 - x_1 x_3$

$+ x_2(x_1 - x_3) - x_1 x_3$

4. (25%) [Diagonal Matrix]

Let $P_2(\mathcal{R})$ be the set of all polynomials with degree less than or equal 2 and with coefficients from a real field $\mathcal{R}$. Let a linear operator $\hat{T} : P_2(\mathcal{R}) \to P_2(\mathcal{R})$ be defined by $\hat{T}(f) = f(0) + f(1)(x + x^2)$.

(a) (10%) Find a basis $\beta$ such that $\left[\hat{T}\right]_\beta$ is a diagonal matrix.

(b) (5%) Show the diagonal matrix.

(c) (10%) Evaluate

$$tr\left[\cos(\alpha\left[\hat{T}\right]_\beta)\right], \tag{3}$$

with the constant $\alpha$. Here, $tr$ denotes trace.

$\lambda - 2\lambda + \lambda^2$

$\lambda(\lambda - 2)$

$d^2 + 6d + 9 - d^2 - 9 - 6d = 0$

$(-3-d)^2 - d^2 + 3(-3-2d) = 0$

$a^2 - d^2 + 3(a-d) = 0$

$\begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 & -1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{pmatrix} 2 & -1 & -2 \end{pmatrix}$

$\left[\begin{array}{ccc|ccc} -1 & -1 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$\begin{array}{ccc|ccc} -1 & 1 & -1 & -1 & 0 & 0 \\ 0 & 1 & 0 & & & \\ 0 & 0 & 1 & & & \end{array}$ $\quad -\frac{1}{3}\ \frac{2}{3}\ -\frac{1}{3}$, $\ -\frac{1}{3}\ -\frac{1}{3}\ \frac{2}{3}$, $\ \frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}$

$bc + d^2 + 3d + 2 = 0$

$ac + cd + 3c = 0, \ c \in R$

$a^2 + bc + 3a + 2 = 0$

$ab + bd + 3b + 0 = 0, \ b \in R$

5. (25%) [Cayley-Hamilton Theorem]

Suppose that a $2 \times 2$ matrix $\overline{\overline{\mathbf{M}}}$ satisfies

$$\overline{\overline{\mathbf{M}}}^2 + 3\overline{\overline{\mathbf{M}}} + 2\overline{\overline{\mathbf{I}}} = \overline{\overline{\mathbf{O}}}, \tag{4}$$

where $\overline{\overline{\mathbf{I}}}$ is a $2 \times 2$ identity matrix and $\overline{\overline{\mathbf{O}}}$ is a $2 \times 2$ zero matrix.

(a) (10%) Determine the eigenvalues of $\overline{\overline{\mathbf{M}}}$.

(b) (10%) Is $\overline{\overline{\mathbf{M}}}^{-1}$ diagonalizable? If yes, find $\overline{\overline{\mathbf{M}}}^{-1}$; If not, explain your answer.

(c) (5%) Calculate $\overline{\overline{\mathbf{M}}}^{(-2)}$ with the help of Cayley-Hamilton theorem.

$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$(a-t)(d-t) - bc = (t^2 + 3t + 2)$

$t^2 - (a+d)t + ad - bc =$

$a + d = -3$

$ad - bc = 2$

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \end{bmatrix} \underbrace{\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}}_{A} \begin{bmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$\overline{\overline{M}} = \begin{bmatrix} (-1)^2 & 0 \\ 0 & (-2)^2 \end{bmatrix} + 3\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$

6. (20%) [Square Root of Matrix]

$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$

Let

$$\overline{\overline{A}} = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}, \tag{5}$$

$(2-\lambda)^3 + \ -3(2-\lambda) + 2$

find the matrix $\overline{\overline{S}}$ such that $\overline{\overline{S}} \cdot \overline{\overline{S}} = \overline{\overline{A}}$.

$(2-\lambda)((2-\lambda)^2$

$\begin{bmatrix} \sqrt{2} & 1 & 1 \\ -1 & -\sqrt{2} & 1 \\ 1 & 1 & -\sqrt{2} \end{bmatrix} \begin{bmatrix} \sqrt{2} & 1 & 1 \\ -1 & \sqrt{2} & 1 \\ 1 & 1 & -\sqrt{2} \end{bmatrix}$

$8 + 12(-\lambda) + 6\lambda^2 - \lambda^3 - 6 + 3\lambda + 2$

$+\lambda^3 + 6\lambda^2 + 9\lambda + 4$

$(\lambda - 1)(\lambda^2 - 5\lambda + 4)$

$(\lambda - 1)(\lambda - 4)$

$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$

$x_1 + x_2 + x_3 = 0$

$x_1 = -x_2 - x_3$

$V_1 = x_2\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & -2 \\ 0 & -1 & 2 \end{bmatrix}$

$\left[\begin{array}{ccc|c} \cancel{-2} & \cancel{1} & \cancel{1} & \cancel{0} \\ 1 & -2 & 1 & 0 \\ 1 & 1 & -2 & 0 \end{array}\right] \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & -2 & 1 \\ 0 & 3 & 3 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad V_2 = t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$1 \ -1$