10806107 林 [illegible]

Linear Algebra, EE 10810EECS205004

First Exam (10:10 AM - 1:00 PM, Friday, October 30th, 2020)

(Dated: Fall, 2020)

Total scores: 120

1. (20%) [**Vector Space and Subspace**]
   Let $\mathcal{W}$ and $\mathcal{U}$ be subspace of a vector space $\mathcal{V}$.
   (a) (5%) Prove that the vector $\vec{0} \in \mathcal{V}$ is *unique*.
   (b) (5%) Prove that for each element $\vec{x} \in \mathcal{V}$, there exists a *unique* vector $\vec{y}$, such that $\vec{x} + \vec{y} = \vec{0}$.
   (c) (5%) Is it possible that $\mathcal{W} \cap \mathcal{U} = \phi$, i.e., ϕ denotes null. ? Why ?
   (d) (5%) Show that $\mathcal{W} \cap \mathcal{U}$ is also a subspace of $\mathcal{V}$.

2. (20%) [**Basis and Dimension**]
   Let S be the set of all positive real numbers. Now, we want to make S as a vector space in $\mathcal{V}$ by asking the following definitions for vectors, vector addition and scalar multiplication:
   - Each element of S will be considered as a "vector" in $\mathcal{V}$.
   - For $A, B \in S$, a "vector sum" is defined as
   $$A + B \equiv AB, \tag{1}$$
   where the product on the right is the usual product of two real numbers.
   - For $c \in \mathcal{R}$(real), and $A \in S$, a "scalar multiplication" is defined as
   $$c \cdot A \equiv A^c, \tag{2}$$
   that is the real number A raised to the c power.

   Now the questions are
   (a) (5%) What is the zero vector in $\mathcal{V}$?
   (b) (10%) Give an example of a set of basis vectors for $\mathcal{V}$.
   (c) (5%) What is the dimension of $\mathcal{V}$?

3. (10%) [**Invertibility**]
   Let $\hat{T} : \mathcal{V} \to \mathcal{V}$ be a linear transformation and $\hat{T}^3(\vec{v}) = 0$ for all $\vec{v} \in \mathcal{V}$. Find the inverse of $\hat{I} - \hat{T}$, where $\hat{I}$ is the identity transformation.

4. (15%) [**Linear Transformation**]
   Let $\overline{\overline{\mathbf{M}}}_{2\times2}(\mathcal{R})$ be the set of 2×2 real matrices. Let $\hat{T} : \overline{\overline{\mathbf{M}}}_{2\times2}(\mathcal{R}) \to \overline{\overline{\mathbf{M}}}_{2\times2}(\mathcal{R})$ be the mapping defined by $\hat{T}(\overline{\overline{\mathbf{A}}}) = \overline{\overline{\mathbf{B}}}\,\overline{\overline{\mathbf{A}}}$, where $\overline{\overline{\mathbf{B}}} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$ and $\overline{\overline{\mathbf{A}}} \in \overline{\overline{\mathbf{M}}}_{2\times2}(\mathcal{R})$.
   (a) (5%) Verify that $\hat{T}$ is a linear operator on $\overline{\overline{\mathbf{M}}}_{2\times2}(\mathcal{R})$.
   (b) (5%) Find the range space of $\hat{T}$, i.e., $R(\hat{T})$.
   (c) (5%) Find the nullspace of $\hat{T}$, i.e., $N(\hat{T})$.

5. (10%) Write down the **Dimension Theorem**, and Explain its meaning in your words.

6. (25%) [**Inverse and Transpose**]
Let $\bar{\bar{\mathbf{A}}}$ be a matrix with the property that

$$\bar{\bar{\mathbf{A}}}^{-1} \equiv c\,\bar{\bar{\mathbf{A}}}^T. \tag{3}$$

$\frac{1}{25}$

   (a) (5%) Show that the matrix $\bar{\bar{\mathbf{B}}}$ defined as

$$\bar{\bar{\mathbf{B}}} = \begin{bmatrix} \bar{\bar{\mathbf{A}}} & \bar{\bar{\mathbf{A}}} \\ -\bar{\bar{\mathbf{A}}} & \bar{\bar{\mathbf{A}}} \end{bmatrix}, \tag{4}$$

   has the inverse in the form

$$\bar{\bar{\mathbf{B}}}^{-1} = d\,\bar{\bar{\mathbf{B}}}^T. \tag{5}$$

$\frac{1}{50}$

   (b) (5%) Find the formula for d.

   (c) (5%) Find $\bar{\bar{\mathbf{B}}}^{-1}$ for the case

$$\bar{\bar{\mathbf{A}}} = \begin{bmatrix} 4 & 3 \\ -3 & 4 \end{bmatrix}. \tag{6}$$

   B^{-1}

   (d) (5%) Find the inverse for the cases in which $\bar{\bar{\mathbf{A}}}$ is the 1×1 matrix, i.e., $\bar{\bar{\mathbf{A}}} = [1]$.

   (e) (5%) Find a formula for the inverse of the matrix

$$\bar{\bar{\mathbf{C}}} = \begin{bmatrix} \bar{\bar{\mathbf{B}}} & \bar{\bar{\mathbf{B}}} \\ -\bar{\bar{\mathbf{B}}} & \bar{\bar{\mathbf{B}}} \end{bmatrix}. \tag{7}$$

7. (20%) [**Rotation**]

   (a) (5%) In 2D, FIG. 1 (Left), Show that the matrix $\overline{\overline{R}}_2$ rotates the column vectors, counterclockwise, with an angle θ has the form:

$$\overline{\overline{R}}_2 = [\hat{I}_{\mathcal{R}^2}]^{\beta}_{\beta'} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}. \tag{8}$$

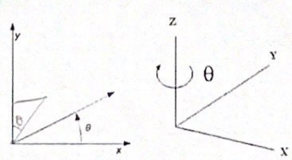

FIG. 1:

   (b) (5%) In 3D, FIG. 1 (Right), by considering the rotation about the $\hat{z}$-axis, but **clockwise**, Construct the corresponding rotation matrix, $\overline{\overline{R}}_3$,

$$\overline{\overline{R}}_3 = [\hat{I}_{\mathcal{R}^3}]^{\beta}_{\beta'} = \begin{bmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{bmatrix}. \tag{9}$$

   (c) (10%) Now, instead, Find the matrix of the rotation by 90°, counterclockwise, about the line spanned by the vector $\vec{c} = (1, 2, 2)$. *Hint:* you may try vectors $\vec{a} = (-2, -1, 2), \vec{b} = (2, -2, 1)$, and $\vec{c}$ as the bases for β'.