Linear Algebra, EE 10810/EECS 205004

Note 2.6 – 3.2

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- 2nd Exam on Dec. xxth, (10:10AM - 1:00 PM, Friday), covering Chap. 2.6, Chap. 3, Chap. 4, Chap 5.

- **Solutions for 1st Exam**:
  - 1(c): Not possible.
  - 2(a): $\{1\}$
  - 2(b): $\{R\}, R > 0 \,\&\, R \neq 1$; any positive real number, but not 1.
  - 2(c): $dim(\mathcal{V}) = 1$
  - 3: $(\hat{I} - \hat{T})^{-1} = \hat{I} + \hat{T} + \hat{T}^2$
  - 4(b): $R(\hat{T}) = \text{span}\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\}$
  - 4(c): $N(\hat{T}) = \text{span}\{\begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}\}$
  - 6(b): $d = \frac{c}{2}$
  - 6(c): $\bar{\bar{\mathbf{B}}}^{-1} = \frac{1}{50}\begin{bmatrix} 4 & -3 & -4 & 3 \\ 3 & 4 & -3 & -4 \\ 4 & -3 & 4 & -3 \\ 4 & 4 & 3 & 4 \end{bmatrix}$
  - 6(d): $\bar{\bar{\mathbf{B}}}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$
  - 6(e): $\bar{\bar{\mathbf{C}}}^{-1} = \frac{c}{4}\bar{\bar{\mathbf{C}}}^{T}$
  - 7(b): $\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$
  - 7(c): $\overline{\overline{R}}^{c}_3 = \left(\begin{bmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{bmatrix}\right)^{-1} \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & 1 \\ -1 & -2 & 2 \\ 2 & 1 & 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 4 & -4 & 7 \\ 8 & 1 & -4 \\ 1 & 8 & 4 \end{bmatrix}$

- **Assignment**:

  1. Prove that $\overline{\overline{E}}$ is an elementary matrix if and only if $\overline{\overline{E}}^{t}$ is.

  2. Find the rank of the following matrices:

$$(a)\begin{pmatrix} 1 & 0 & 2 \\ 1 & 1 & 4 \end{pmatrix} \qquad (b)\begin{pmatrix} 1 & 2 & 0 & 1 & 1 \\ 2 & 4 & 1 & 3 & 0 \\ 3 & 6 & 2 & 5 & 1 \\ -4 & -8 & 1 & -3 & 1 \end{pmatrix} \tag{1}$$

From Scratch !!

- Definition: the **dual space** of $\mathcal{V}$ is the vector space $\mathcal{L}(\mathcal{V}, F)$, denoted by $\mathcal{V}^*$.
- Theorem 2.24: Let $\beta = \{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n\}$ be the ordered basis of $\mathcal{V}$, and we can find $\beta^* = \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\}$ as an ordered basis for $\mathcal{V}^*$, for any $f \in \mathcal{V}^*$, we have

$$f = \sum_{i=1}^{n} f(\vec{x}_i)\, f_i. \tag{2}$$

- Definition: the ordered bases $\beta^* = \{\vec{f}_1, \vec{f}_2, \ldots, \vec{f}_n\}$ of $\mathcal{V}^*$ that satisfies $f_i(\vec{x}_j) = \delta_{ij}$ is called the **dual basis** of β.
- Theorem 2.25: for any linear transformation $\hat{T} : \mathcal{V} \to \mathcal{W}$, the mapping $\hat{T}^t : \mathcal{W}^* \to \mathcal{V}^*$ defінted by $\hat{T}^t(g) = g\hat{T}$ for all $g \in \mathcal{W}^*$ is a linear transformation with the property hat

$$[\hat{T}^t]_{\gamma^*}^{\beta^*} = ([\hat{T}]_\beta^\gamma)^t \tag{3}$$

- Definition: for a $\vec{x} \in \mathcal{V}$, the **linear functional** on $\mathcal{V}^*$ is defined as $\hat{x} : \mathcal{V}^* \to F$ by $\hat{x} = f(x)$.
- Lemma: If $\hat{x}(f) = 0$ for all $f \in \mathcal{V}^*$, then $\vec{x} = 0$.
- Theorem 2.26: $\psi : \mathcal{V} \to \mathcal{V}^{**}$ by $\psi(\vec{x}) = \hat{x}$ is an isomorphism.
- Corollary: every ordered basis for $\mathcal{V}^*$ is the dual basis for some basis for $\mathcal{V}$.

- Section 2.7: Homogeneous Linear Differential Equations with constant coefficients

- Definition: Elementary row [column] operations:
- Type 1, 2, and 3 elementary matrix: $\overline{\overline{P}}$, $\overline{\overline{D}}$, and $\overline{\overline{E}}$
- Theorem 3.1: There exists an $m \times m$ $(n \times n)$ elementary matrix $\overline{\overline{E}}$, such that $\overline{\overline{B}} = \overline{\overline{EA}}_{m\times n}$ (or $\overline{\overline{B}} = \overline{\overline{A}}_{m\times n}\overline{\overline{E}}$)
- Theorem 3.2: Elementary matrices are invertible, and the inverse oof an elementary matrix is an elementary matrix of the same type.

- Definition: If $\overline{\overline{A}} \in \overline{\overline{M}}_{m\times n}(F)$, the **rank** of $\overline{\overline{A}}$, denoted $rank(\overline{\overline{A}})$, is the rank of the linear transformation $\hat{L}_A : F^n \to F^m$.
- Corollary of Theorem 2.18: an $n \times n$ matrix is invertible if and only if its rank is n.
- Theorem 3.3: $rank(\hat{T}) = rank([\hat{T}]_\beta^\gamma)$
- Theorem 3.4: If $\overline{\overline{P}}_{m\times m}$ and $\overline{\overline{Q}}_{n\times n}$ are invertible matrices, then
  1. $rank(\overline{\overline{A}}_{m\times n}\overline{\overline{Q}}) = rank(\overline{\overline{A}})$,
  2. $rank(\overline{\overline{PA}}_{m\times n}) = rank(\overline{\overline{A}})$,
  3. $rank(\overline{\overline{PA}}_{m\times n}\overline{\overline{Q}}) = rank(\overline{\overline{A}})$,
- Corollary: Elementary row and column opeartion on a matrix are *rank-preserving*.
- Theorem 3.5: The rank of any matrix equals the maximum number of its linearly independent columns;
- Theorem 3.5: The rank of a matrix is the dimension of the subspace generated by its columns.
- Theorem 3.6: Let $\overline{\overline{A}}_{m\times n}$ has the rank r. Then $r \le m$, $r \le n$, and $\overline{\overline{A}}$ can be transformed into

$$\overline{\overline{D}} = \begin{pmatrix} \overline{\overline{I}}_r & \overline{\overline{O}}_1 \\ \overline{\overline{O}}_2 & \overline{\overline{O}}_3 \end{pmatrix} \tag{4}$$

- Corollay 2: $rank(\overline{\overline{A}}^t) = rank(\overline{\overline{A}})$