

EE 2030 Linear Algebra

Homework #6

Due to 06/07/2023

1.

Suppose G_{k+2} is the *average* of the two previous numbers G_{k+1} and G_k :

$$G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \quad \text{is} \quad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = [A] \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}.$$

$$G_{k+1} = G_{k+1}$$

(a) Find the eigenvalues and eigenvectors of A .

(b) Find the limit as $n \rightarrow \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

(c) If $G_0 = 0$ and $G_1 = 1$ show that the Gibonacci numbers approach $\frac{2}{3}$.

2.

The eigenvalues of A are 1 and 9, and the eigenvalues of B are -1 and 9:

$$A = \begin{bmatrix} 5 & 4 \\ 4 & 5 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 4 & 5 \\ 5 & 4 \end{bmatrix}.$$

Find a matrix square root of A from $R = S\sqrt{\Lambda}S^{-1}$. Why is there no real matrix square root of B ?

3.

A door is opened between rooms that hold $v(0) = 30$ people and $\omega(0) = 10$ people. The movement between rooms is proportional to the difference $v - \omega$:

$$\frac{dv}{dt} = \omega - v \quad \text{and} \quad \frac{d\omega}{dt} = v - \omega.$$

Show that the total $v + \omega$ is constant (40 people). Find the matrix in $\frac{du}{dt} = Au$

And its eigenvalues and eigenvectors. What are v and ω at $t = 1$ and $t = \infty$?

4.

Write $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix}$ as $S\Lambda S^{-1}$. Multiply $Se^{At}S^{-1}$ to find the matrix exponential e^{At} . Check e^{At} and the derivative of e^{At} when $t = 0$.

5.

(Recommended) This matrix M is skew-symmetric and also _____. Then all its eigenvalues are pure imaginary and they also have $|\lambda| = 1$. ($\|Mx\| = \|x\|$ for every x so $\|\lambda x\| = \|x\|$ for eigenvectors.) Find all four eigenvalues from the trace of M :

$$M = \frac{1}{\sqrt{3}} \begin{bmatrix} 0 & 1 & 1 & 1 \\ -1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 \\ -1 & -1 & 1 & 0 \end{bmatrix} \quad \text{can only have eigenvalues } i \text{ or } -i.$$

6.

(Recommended) Which of these classes of matrices do A and B belong to:

Invertible, orthogonal, projection, permutation, diagonalizable, Markov?

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad B = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Which of these factorizations are possible for A and B : LU, QR, SAS^{-1}, QAQ^T ?

7.

Which 3 by 3 symmetric matrices A and B produce these quadratics?

$$x^T A x = 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_2 x_3). \text{ Why is } A \text{ positive definite?}$$
$$x^T B x = 2(x_1^2 + x_2^2 + x_3^2 - x_1 x_2 - x_1 x_3 - x_2 x_3). \text{ Why is } B \text{ positive semidefinite?}$$

8.

For which s and t do A and B have all $\lambda > 0$ (therefore positive definite)?

$$A = \begin{bmatrix} s & -4 & -4 \\ -4 & s & -4 \\ -4 & -4 & s \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} t & 3 & 0 \\ 3 & t & 4 \\ 0 & 4 & t \end{bmatrix}.$$

9.

There are sixteen 2 by 2 matrices whose entries are 0's and 1's. Similar matrices go into the same family. How many families? How many matrices (total 16) in each family?

10.

These Jordan matrices have eigenvalues 0, 0, 0, 0. They have two eigenvectors (one from each block). But the block size don't match and they are not *similar*.

$$J = \left[\begin{array}{ccc|cc} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 \end{array} \right] \quad \text{and} \quad K = \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array} \right].$$

For any matrix M , compare JM with MK . If they are equal show that M is not invertible. Then $M^{-1}JM = K$ is impossible: J is not similar to K .

11.

Find the eigenvalues and unit eigenvectors v_1, v_2 of $A^T A$. Then find $u_1 = Av_1/\sigma_1$:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \quad \text{and} \quad A^T A = \begin{bmatrix} 10 & 20 \\ 20 & 40 \end{bmatrix} \quad \text{and} \quad AA^T = \begin{bmatrix} 5 & 15 \\ 15 & 45 \end{bmatrix}.$$

Verify that u_1 is a unit eigenvector of AA^T . Complete the matrices U, Σ, V .

$$\text{SVD} \quad \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = [u_1 \ u_2] \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} [v_1 \ v_2]^T.$$

12.

Compute $A^T A$ and AA^T and their eigenvalues and unit eigenvectors for V and U .

$$\text{Rectangular matrix} \quad A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}.$$

Check $AV = U\Sigma$ (this will decide \pm signs in U). Σ has the same shape as A .