

# EE2030 Linear Algebra

## Homework #3

Due: 03/29/2023 10:10(Wed)

1. What conditions on  $b_1, b_2, b_3, b_4$  make each system solvable? Find  $\mathbf{x}$  in that case:

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \\ 2 & 5 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 2 & 5 & 7 \\ 3 & 9 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$$

2. Find by elimination the rank of  $A$  and also the rank of  $A^T$ :

$$A = \begin{bmatrix} 1 & 4 & 0 \\ 2 & 11 & 5 \\ -1 & 2 & 10 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & q \end{bmatrix} \quad (\text{rank depends on } q)$$

3. Reduce to  $U\mathbf{x} = \mathbf{c}$  (Gaussian elimination) and then  $R\mathbf{x} = \mathbf{d}$  (Gauss-Jordan):

$$A\mathbf{x} = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 3 & 2 & 0 \\ 2 & 0 & 4 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix} = \mathbf{b}.$$

Find a particular solution  $\mathbf{x}_p$  and all homogeneous solutions  $\mathbf{x}_n$ .

4.  $U$  comes from  $A$  by subtracting row 1 from row 3:

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 1 & 3 & 2 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Find bases for the two column spaces. Find bases for the two row spaces. Find bases for the two nullspaces. Which spaces stay fixed in elimination?

5. For which numbers  $c$  and  $d$  do these matrices have rank 2?

$$A = \begin{bmatrix} 1 & 2 & 5 & 0 & 5 \\ 0 & 0 & c & 2 & 2 \\ 0 & 0 & 0 & d & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$$

6. Choose  $\mathbf{x} = (x_1, x_2, x_3, x_4)$  in  $\mathbf{R}^4$ . It has 24 rearrangements like  $(x_2, x_1, x_3, x_4)$  and  $(x_4, x_3, x_1, x_2)$ . Those 24 vectors, including  $\mathbf{x}$  itself, span a subspace  $\mathbf{S}$ . Find specific vectors  $\mathbf{x}$  so that the dimension of  $\mathbf{S}$  is: (a) zero, (b) one, (c) three, (d) four.

7. Without computing  $A$ , find bases for its four fundamental subspaces:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 6 & 1 & 0 \\ 9 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$$

8. (Left nullspace) Add the extra column  $\mathbf{b}$  and reduce  $A$  to echelon form:

$$[A \quad \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 4 & 5 & 6 & b_2 \\ 7 & 8 & 9 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & 3 & b_1 \\ 0 & -3 & -6 & b_2 - 4b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + b_1 \end{bmatrix}$$

A combination of the rows of  $A$  has produced the zero row. What combination is it? (Look at  $b_3 - 2b_2 + b_1$  on the right side.) Which vectors are in the nullspace of  $A^T$  and which vectors are in the nullspace of  $A$ ?

9.  $\mathbf{M}$  is the space of 3 by 3 matrices. Multiply every matrix  $X$  in  $\mathbf{M}$  by

$$A = \begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}. \quad \text{Notice: } A \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

(a) which matrices  $X$  leads to  $AX =$  zero matrix?

(b) which matrices have the form  $AX$  for some matrix  $X$ ?

(a) finds the "nullspace" of that operation  $AX$  and (b) finds the "column space". What are the dimensions of those two subspaces of  $\mathbf{M}$ ? Why do the dimensions add to  $(n - r) + r = 9$ ?