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1.

Assume $\vec{v} = Ax = B\hat{x}$ is in $C(A)$ and $C(B)$, $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, $\hat{x} = \begin{bmatrix} x_3 \\ x_4 \end{bmatrix}$

$\rightarrow [A\ -B] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = Ax - B\hat{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

By Gauss-Jordan method, $\left[\begin{array}{cccc|c} 1 & 2 & -5 & -4 & 0 \\ 1 & 3 & -6 & -3 & 0 \\ 1 & 2 & -5 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & -5 & -4 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 0 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 0 & -3 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = 3x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + x \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix} = x \begin{bmatrix} 5 \\ 6 \\ 5 \end{bmatrix}$

2.(a)

$\forall v \in S^\perp$ s.t. $v^T \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow S^\perp = R^3$

2.(b)

$S = \{k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \mid k \in R\}$, $\forall v \in S^\perp$ s.t. $v^T (k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow S^\perp = \{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}\}$

2.(c)

$S = \{m \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \mid m, n \in R\}$, $\forall v \in S^\perp$ s.t. $v^T (m \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + n \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$S^\perp = \{k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \mid k \in R\}$, the basis of $S^\perp$ can be $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$

3.(a)

False. Two plane with perpendicular normal vectors must share a nonzero vector, so they can't be orthogonal spaces.

3.(b)

False. Assume v in the first subspace and w in the second space,

$v = a \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$, $w = c \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} + d \begin{bmatrix} 2 \\ -2 \\ 3 \\ 4 \end{bmatrix}$, $a \cdot b \cdot c \cdot d \in R$

$\because v^T w = ac \cdot 0 + ad \cdot 0 + bc \cdot 0 + bd \cdot 3$ might not be zero

$\therefore$ The first space isn't the orthogonal complement of the second one

3.(c)

False. Let $A = \{a \begin{bmatrix} 1 \\ 1 \end{bmatrix} \mid a \in R\}$ and $B = \{b \begin{bmatrix} 1 \\ 2 \end{bmatrix} \mid a \in R\}$, $v \in A$ and $w \in B$

$v^T w = ab \cdot 3$ might not be zero, so A and B aren't orthogonal

4.

$C(A) = \{x \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \mid x, y, z \in R\}$, the projection of b is $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix}$

$P \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \rightarrow P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

5.

$\because P^2 = P$ and P project onto A

$\therefore \forall v \in C(A)$ and x s.t. $Px = v$, $(I - P)x = x - Px = x - v$

$\because v$ is the projection of x and $v \in C(A)$

$\therefore (x - v) \perp v$, $(x - v) \in N(A^T)$

6.

$\because P^T = P$ and $P^2 = P$ $\quad \therefore P^T P = P^2 = P$

P_{22} = The second row of P^T x The second column of $P = P_{21} \cdot P_{12} + P_{22}^2 + P_{23} \cdot P_{32}$

$\because P^T = P$ $\quad \therefore P_{21} = P_{12}$, $P_{23} = P_{32}$, $P = P_{21}^2 + P_{22}^2 + P_{23}^2$

7.

For the best line $C + Dt = b$ and $A = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix}$, $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ s.t. $A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^T b$

$\because A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ t_1 & t_2 & t_3 & t_4 \end{bmatrix} \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ 1 & t_3 \\ 1 & t_4 \end{bmatrix} = \begin{bmatrix} 4 & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix}$, $A^T b = \begin{bmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{bmatrix}$ $\therefore \begin{bmatrix} 4 & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \Sigma b_i \\ \Sigma t_i b_i \end{bmatrix}$,

and there is $[4\ \Sigma t_i] \begin{bmatrix} C \\ D \end{bmatrix} = \Sigma b_i \rightarrow C + D\frac{\Sigma t_i}{4} = \frac{\Sigma b_i}{4}$

so the line $C + Dt = b$ must pass through $(\frac{1}{4}\Sigma t_i, \frac{1}{4}\Sigma b_i) = (2, 9)$

8.(a)

$\because a = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$, $a^T a = m$, $a^T b = \Sigma b_i$ $\quad \therefore m\hat{x} = \Sigma b_i$, $\hat{x} = \frac{\Sigma b_i}{m} = \bar{b}$

8.(b)

From 8.(a), $e = b - a\hat{x} = \begin{bmatrix} b_1 - \bar{b} \\ b_2 - \bar{b} \\ \vdots \\ b_m - \bar{b} \end{bmatrix}$, $\|b\|^2 = \sum_{i=1}^{m} (b_i - \bar{b})^2$, $\|e\| = \sqrt{\sum_{i=1}^{m} (b_i - \bar{b})^2}$

8.(c)

$e = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$, $p^T e = [2, 3, 3] \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix} = 0 \rightarrow p \perp e$

$P = \frac{AA^T}{A^T A} = \frac{1}{3} AA^T = \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

9.

$A = [1\ t]$ statifies that $A^T A \begin{bmatrix} C \\ D \end{bmatrix} = A^T b$

$\begin{bmatrix} 5 & \Sigma t_i \\ \Sigma t_i & \Sigma t_i^2 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} \Sigma b_i \\ \Sigma b_i t_i \end{bmatrix}$, $\begin{bmatrix} 5 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 5 \\ -10 \end{bmatrix}$, $\begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $C + Dt = 1 - t$

10.

First, $A = a = (1, -1, 0, 0)$

Second, $B = b - \frac{A^T b}{A^T A} A = b + \frac{1}{2} A = (\frac{1}{2}, \frac{1}{2}, -1, 0)$

Third, $C = c - \frac{A^T c}{A^T A} A - \frac{B^T c}{B^T B} B = C + \frac{2}{3} B = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, -1)$

11.(a)

$x_4 = x_1 + x_2 + x_3$, the solution is $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$,

so the basis of S can be $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

11.(b)

$\forall k = \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} \in S^\perp$, $k \perp x \rightarrow k$ is perpendicular to the three basis of S

$\rightarrow$ For basis of S, $b_1 b_2$ and b_3, $k^T b_1 = k^T b_2 = k^T b_3 = 0$

$k_1 + k_4 = 0$
$k_2 + k_4 = 0$
$k_3 + k_4 = 0$
$\rightarrow \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = k_4 \begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$, the basis of $S^\perp$ can be $\begin{bmatrix} -1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

11.(c)

$b_1 = \text{proj}_S(b)$, $b_2 = b - b_1$, $S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

$S^T S \hat{x} = S^T b$, $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \hat{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$, $b_1 = S\hat{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} (\frac{1}{4} \begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}) = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$

$b_2 = b - b_1 = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -\frac{1}{2})$

12.(a)

$\|q_i\|^2 = 4c^2 \rightarrow$ If Q is an orthogonal matrix, $4c^2 = 1$ and $c = \pm\frac{1}{2}$

12.(b)

$\text{proj}_{q_1}(b) = \frac{q_1^T b}{q_1^T q_1} q_1 = q_1 = (1, -1, -1, -1)$

Let $Q = [q_1\ q_2]$, $Q^T Q \hat{x} = Q^T b$

$\because \|q_1\| = \|q_2\| = 1$ $\quad \therefore \hat{x} = Q^T b$, $p = Q\hat{x} = QQ^T b = \frac{1}{4} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ -1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 & -1 & -1 \\ -1 & 1 & -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

projection $= (0, 0, 1, 1)$

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