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1.(a)

By $\begin{cases} G_{k+2} = \frac{1}{2}G_{k+1} + \frac{1}{2}G_k \\ G_{k+1} = G_{k+1} \end{cases}$, $A = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{bmatrix}$

$\det(A-\lambda I) = (\frac{1}{2}-\lambda)(-\lambda) - \frac{1}{2}\times 1 = 0$, $\lambda = 1$ or $-\frac{1}{2}$

If $Ax = 1\cdot x$, $x = k\begin{bmatrix}1\\1\end{bmatrix}$, $k \in R$

If $Ax = -\frac{1}{2}x$, $x = k\begin{bmatrix}1\\-2\end{bmatrix}$, $k \in R$ $\Rightarrow$ The eigenvectors are $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-2\end{bmatrix}$

1.(b)

$$\lim_{n\to\infty} S\Lambda^n S^{-1} = \lim_{n\to\infty}\begin{bmatrix}1&1\\1&-2\end{bmatrix}\begin{bmatrix}(1)^n&0\\0&(-\frac{1}{2})^n\end{bmatrix}\begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{1}{3}&-\frac{1}{3}\end{bmatrix} = \begin{bmatrix}\frac{2}{3}&\frac{1}{3}\\\frac{2}{3}&\frac{1}{3}\end{bmatrix}$$

1.(c)

$$\begin{bmatrix}G_1\\G_0\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} = \frac{2}{3}\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-2\end{bmatrix}$$

$$\lim_{k\to\infty}\begin{bmatrix}G_{k+1}\\G_k\end{bmatrix} = \lim_{k\to\infty}A^k\begin{bmatrix}G_1\\G_0\end{bmatrix} = \lim_{k\to\infty}\left(\frac{2}{3}\cdot 1^k\begin{bmatrix}1\\1\end{bmatrix} + \frac{1}{3}\cdot(-\frac{1}{2})^k\begin{bmatrix}1\\-2\end{bmatrix}\right) = \frac{2}{3}\begin{bmatrix}1\\1\end{bmatrix}$$

$$\lim_{k\to\infty} G_k = \frac{2}{3}\times 1 = \frac{2}{3}$$

2.(a)

By the eigenvalues of A, the eigenvectors are $\begin{bmatrix}1\\-1\end{bmatrix}$ and $\begin{bmatrix}1\\1\end{bmatrix}$

$A = \begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}1&0\\0&9\end{bmatrix}\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$

R has the same eigenvectors and the square root of eigenvalues from A

$R = \begin{bmatrix}1&1\\-1&1\end{bmatrix}\begin{bmatrix}\pm 1&0\\0&\pm 3\end{bmatrix}\begin{bmatrix}\frac{1}{2}&-\frac{1}{2}\\\frac{1}{2}&\frac{1}{2}\end{bmatrix}$

2.(b)

Because the square root of -1 is imaginary number, the square root of B isn't real

3.(a)

$\frac{d}{dt}v + \frac{d}{dt}w = \frac{d}{dt}(v+w) = 0$, so $v+w$ is a constant

3.(b)

By the two equation, if $\frac{d}{dt}\begin{bmatrix}v\\w\end{bmatrix} = A\begin{bmatrix}v\\w\end{bmatrix}$, $A\begin{bmatrix}v\\w\end{bmatrix} = \begin{bmatrix}-1&1\\1&-1\end{bmatrix}\cdot\begin{bmatrix}v\\w\end{bmatrix}$, $A = \begin{bmatrix}-1&1\\1&-1\end{bmatrix}$

3.(c)

$\because \frac{d}{dt}(v+w) = (w-v) + (v-w) = 0$, $\frac{d}{dt}(v-w) = -2(v-w)$

$\therefore$ The eigenvalue is 0 and -2, the eigenvector is $\begin{bmatrix}1\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$

3.(d)

By the answer of 2.(c), the solution of $\begin{bmatrix}v\\w\end{bmatrix}$ is $Ce^{0t}\begin{bmatrix}1\\1\end{bmatrix} + De^{-2t}\begin{bmatrix}1\\-1\end{bmatrix}$

When $t=0$, $\begin{bmatrix}v\\w\end{bmatrix} = C\begin{bmatrix}1\\1\end{bmatrix} + D\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}30\\10\end{bmatrix}$, $C=20$, $D=10$

When $t=1$, $\begin{bmatrix}v\\w\end{bmatrix} = 20\begin{bmatrix}1\\1\end{bmatrix} + 10e^{-2}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}20+10e^{-2}\\20-10e^{-2}\end{bmatrix}$

When $t\to\infty$, $\begin{bmatrix}v\\w\end{bmatrix} = 20\begin{bmatrix}1\\1\end{bmatrix} + 10e^{-2t}\begin{bmatrix}1\\-1\end{bmatrix} \to \begin{bmatrix}20\\20\end{bmatrix}$

4.

$\det(A-\lambda I) = (1-\lambda)(3-\lambda) - 0 = 0$, $\lambda = 1$ or 3, eigenvectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}1\\2\end{bmatrix}$

$\Lambda = \begin{bmatrix}1&0\\0&3\end{bmatrix}$, $S = \begin{bmatrix}1&1\\0&2\end{bmatrix}$, $A = \begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}1&0\\0&3\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}$

$e^{At} = Se^{\Lambda t}S^{-1} = \begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}e^t&0\\0&e^{3t}\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix}$

When $t=0$, $e^{At} = \begin{bmatrix}1&1\\0&2\end{bmatrix}\begin{bmatrix}1&0\\0&1\end{bmatrix}\begin{bmatrix}1&-\frac{1}{2}\\0&\frac{1}{2}\end{bmatrix} = \begin{bmatrix}1&0\\0&1\end{bmatrix}$

$\frac{d}{dt}e^{At} = Ae^{At} = A = \begin{bmatrix}1&1\\0&3\end{bmatrix}$

5.

M is also orthogonal, so $|\lambda| = 1$

$\because$ The trace $=0$ and $\det M = 1$

$\therefore \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = 0$, $\lambda_1\lambda_2\lambda_3\lambda_4 = 1$

$\lambda_1 \sim \lambda_4$ can only be i or $-i$

6.

A is invertible, and the inverse $A^{-1} = A$

A is orthogonal because each length of cloumn is 1 and the columns are orthogonal

A isn't projection. It doesn't project all vectors to a space

A is permutation. It changes the first and the third rows

A is diagonalizable and $A = \begin{bmatrix}1&1&1\\0&0&1\\-1&1&1\end{bmatrix}\begin{bmatrix}-1&0&0\\0&1&0\\0&0&1\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0&-\frac{1}{2}\\\frac{1}{2}&-1&\frac{1}{2}\\0&1&0\end{bmatrix}$

A is Markov because each entry ≥ 0 and the sum of each column is 1

B isn't invertible, for it's signular

B isn't orthogonal. Each column is not orthogonal with each other

B is projection. It projects vectors onto $x=y=z$

B isn't permutation

B isn't diagonalizabe, for it doesn't have enough independent eigenvectors

B is Markov. Each entry ≥ 0 and the sum of each column $=1$

For A, LU, QR and $S\Lambda S^{-1}$ are possible

For B, these are all impossible

7.(a)

$x_1x_2 \leq (\frac{x_1+x_2}{2})^2 = \frac{1}{4}x_1^2 + \frac{1}{4}x_2^2 + \frac{1}{2}x_1x_2$, $x_1x_2 \leq \frac{1}{2}(x_1^2+x_2^2)$

$x_2x_3 \leq (\frac{x_2+x_3}{2})^2 = \frac{1}{4}x_2^2 + \frac{1}{4}x_3^2 + \frac{1}{2}x_2x_3$, $x_2x_3 \leq \frac{1}{2}(x_2^2+x_3^2)$

$x^TAx = 2(x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3) \geq 2(x_1^2+x_2^2+x_3^2-\frac{1}{2}(x_1^2+x_2^2)-\frac{1}{2}(x_2^2+x_3^2))$
$= x_1^2+x_3^2$

$x^TAx \geq 0$ and may be 0 only when $x_1 = x_3 = 0$

$\because$ If $\exists x \neq \vec{0}$ and $x^TAx = 0$, $x_2 \neq 0$, $x^TAx = 2(x_2^2) = 2x_2^2 \neq 0$ $\divideontimes$

$\therefore \forall x \neq \vec{0}$, $x^TAx > 0$, A is positive definite

7.(b)

By the method in 7(a)

$\because x^TBx \geq (x_1^2+x_2^2+x_3^2 - \frac{1}{2}(x_1^2+x_2^2) - \frac{1}{2}(x_2^2+x_3^2) - \frac{1}{2}(x_1^2+x_3^2)) = 0$

and if $x_1=x_2=x_3$, $x_1x_2 = (\frac{x_1+x_2}{2})^2$, $x_1x_3 = (\frac{x_1+x_3}{2})^2$, $x_2x_3 = (\frac{x_2+x_3}{2})^2$, $x^TBx = 0$

If $x \neq 0$, $x^TBx \geq 0$ and $\exists x$ s.t. $x^TBx = 0$

$\therefore$ B is positive semidefinite

8.

$\det(A-\lambda I) = (s-\lambda)[(s-\lambda)^2-16] + 4[-4(s-\lambda)-16] - 4[16+4(s-\lambda)]$
$= (s-\lambda)^3 - 48(s-\lambda) - 128 = 0$
$\lambda = s-8$ or $s+4$

If $s > 8$, all $\lambda > 0$

$\det(B-\lambda I) = (t-\lambda)[(t-\lambda)^2-16] - 3\times 3(t-\lambda)$
$= (t-\lambda)^3 - 25(t-\lambda)$
$\lambda = t, t-5$ or $t+5$

If $t > 5$, all $\lambda > 0$

9.

Consider a matrix $\begin{bmatrix}a&c\\b&d\end{bmatrix}$

The eigenvalues satisfy $(a-\lambda)(d-\lambda) - bc = 0$, $\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}$

When $a+d = 2$, the discriminant can be 0, 4

When $a+d = 1$, the discriminant can be 1, 5

When $a+d = 0$, the discriminant can be 0, 4

There are 6 kinds of λ, so there are 6 families

The 2/0 family has $(a,b,c,d) = (1,0,0,1), (1,0,1,1)$, and $(1,1,0,1)$

The 2/4 family has $(a,b,c,d) = (1,1,1,1)$

The 1/1 family has $(a,b,c,d) = (1,0,0,0), (1,0,1,0), (1,1,0,0),$
$(0,0,0,1), (0,0,1,1), (0,1,0,1)$

The 1/5 family has $(a,b,c,d) = (1,1,1,0), (0,1,1,1)$

The 0/0 family has $(a,b,c,d) = (0,0,0,0), (0,0,1,0), (0,1,0,0)$

The 0/4 family has $(a,b,c,d) = (0,1,1,0)$

10.

Because all the eigenvalues are 0, any 4×1 $\vec{x}$ has $J\vec{x} = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$, and $JM = \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}$

For $M = [M_1\ M_2\ M_3\ M_4]$, $MK = \begin{bmatrix}\vdots & M_1 & M_2 & \vdots\end{bmatrix}$

If $JM = MK$, $M_1 = M_2 = \begin{bmatrix}0\\0\\0\\0\end{bmatrix}$ and M must be not invertible. J isn't similar to K

11.

$\det(A^TA - \lambda I) = (10-\lambda)(40-\lambda) - 400 = 0 \Rightarrow \lambda = 50$ or 0

$\begin{bmatrix}10&20\\20&40\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 50\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = k\begin{bmatrix}1\\2\end{bmatrix}$ $\quad v_1 = \begin{bmatrix}1\\2\end{bmatrix}/\sqrt{1^2+2^2} = \begin{bmatrix}\frac{1}{\sqrt5}\\\frac{2}{\sqrt5}\end{bmatrix}$

$\begin{bmatrix}10&20\\20&40\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = 0\begin{bmatrix}x\\y\end{bmatrix} \Rightarrow \begin{bmatrix}x\\y\end{bmatrix} = k\begin{bmatrix}-2\\1\end{bmatrix}$ $\quad v_2 = \begin{bmatrix}2\\-1\end{bmatrix}/\sqrt{2^2+(-1)^2} = \begin{bmatrix}\frac{2}{\sqrt5}\\-\frac{1}{\sqrt5}\end{bmatrix}$

$u_1 = \frac{\begin{bmatrix}1&2\\3&6\end{bmatrix}\begin{bmatrix}\frac{1}{\sqrt5}\\\frac{2}{\sqrt5}\end{bmatrix}}{\sqrt{50}} = \begin{bmatrix}\frac{1}{\sqrt{10}}\\\frac{3}{\sqrt{10}}\end{bmatrix}$, $\|u_1\|^2 = \frac{1}{10} + \frac{9}{10} = 1$, $u_2 = \begin{bmatrix}0\\0\end{bmatrix}$

$U = [u_1\ u_2] = \begin{bmatrix}\frac{1}{\sqrt{10}}&0\\\frac{3}{\sqrt{10}}&0\end{bmatrix}$

$\Sigma = \begin{bmatrix}\sqrt{50}&0\\0&0\end{bmatrix}$

$V = \begin{bmatrix}\frac{1}{\sqrt5}&\frac{2}{\sqrt5}\\\frac{2}{\sqrt5}&\frac{-1}{\sqrt5}\end{bmatrix}$

12.

rank $=2$

$A^TA = \begin{bmatrix}1&1&0\\1&2&1\\0&1&1\end{bmatrix}$, $\lambda = 3, 1$, and 0, the eigenvectors are $\frac{1}{\sqrt6}\begin{bmatrix}1\\2\\1\end{bmatrix}$, $\frac{1}{\sqrt2}\begin{bmatrix}-1\\0\\1\end{bmatrix}$, and $\frac{1}{\sqrt3}\begin{bmatrix}1\\-1\\1\end{bmatrix}$

$AA^T = \begin{bmatrix}2&1\\1&2\end{bmatrix}$, $\lambda = 3$ and 1, the eigenvectors are $\frac{1}{\sqrt2}\begin{bmatrix}1\\1\end{bmatrix}$ and $\frac{1}{\sqrt2}\begin{bmatrix}-1\\1\end{bmatrix}$

$V = \begin{bmatrix}\frac{1}{\sqrt6}&-\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\\\frac{2}{\sqrt6}&0&-\frac{1}{\sqrt3}\\\frac{1}{\sqrt6}&\frac{1}{\sqrt2}&\frac{1}{\sqrt3}\end{bmatrix}$, $U = \begin{bmatrix}\frac{1}{\sqrt2}&-\frac{1}{\sqrt2}\\\frac{1}{\sqrt2}&\frac{1}{\sqrt2}\end{bmatrix}$, $\Sigma = \begin{bmatrix}\sqrt3&0&0\\0&1&0\end{bmatrix}$

$AV = \begin{bmatrix}\frac{3}{\sqrt6}&-\frac{1}{\sqrt2}&0\\\frac{3}{\sqrt6}&\frac{1}{\sqrt2}&0\end{bmatrix}$ $\quad AV = U\Sigma$

$U\Sigma = \begin{bmatrix}\frac{\sqrt3}{\sqrt2}&-\frac{1}{\sqrt2}&0\\\frac{\sqrt3}{\sqrt2}&\frac{1}{\sqrt2}&0\end{bmatrix}$

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