1.

In the first equation, $[b] = x_1[A_1] + x_2[A_2]$ for $[A_1] = \begin{bmatrix}1\\2\\2\\3\end{bmatrix}$ and $[A_2] = \begin{bmatrix}2\\4\\5\\9\end{bmatrix}$

In the second equation, $[b] = x_1[A_1] + x_2[A_2] + x_3[A_3]$ for $[A_3] = \begin{bmatrix}3\\6\\7\\12\end{bmatrix}$

If the equations are solvable, $[b]$ must be in $C(A)$

$\because [A_3] = [A_1] + [A_2]$, $[b] = (x_1 + x_3)[A_1] + (x_2 + x_3)[A_2]$

$\therefore$ If the two equations are solvable, $[b]$ must be in $C(A)$ for $A \begin{bmatrix}1&2\\2&4\\2&5\\3&9\end{bmatrix}$ #

2.

For the first matrix,

$$A = \begin{bmatrix}1&4&0\\2&11&5\\-1&2&10\end{bmatrix} \to \begin{bmatrix}1&4&0\\0&3&5\\0&6&10\end{bmatrix} \to \begin{bmatrix}1&4&0\\0&1&\frac{5}{3}\\0&0&0\end{bmatrix}$$, the rank of $A = 2$

$$A^T = \begin{bmatrix}1&2&-1\\4&11&2\\0&5&10\end{bmatrix} \to \begin{bmatrix}1&2&-1\\0&3&6\\0&5&10\end{bmatrix} \to \begin{bmatrix}1&2&-1\\0&1&2\\0&0&0\end{bmatrix}$$, the rank of $A^T = 2$

For the second matrix,

$$A = \begin{bmatrix}1&0&1\\1&1&2\\1&1&q\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&1\\0&1&q-1\end{bmatrix} \to \begin{bmatrix}1&0&1\\0&1&1\\0&0&q-2\end{bmatrix}$$, If $q = 2$, rank $= 2$. If $q \neq 2$, rank $= 3$

$$A^T = \begin{bmatrix}1&1&1\\0&1&1\\1&2&q\end{bmatrix} \to \begin{bmatrix}1&1&1\\0&1&1\\0&1&q-1\end{bmatrix} \to \begin{bmatrix}1&1&1\\0&1&1\\0&0&q-2\end{bmatrix}$$, If $q = 2$, rank $= 2$, If $q \neq 2$, rank $= 3$

3.

$$[A \mid b] = \left[\begin{array}{cccc|c}1&0&2&3&2\\1&3&2&0&5\\2&0&4&9&10\end{array}\right] \to \left[\begin{array}{cccc|c}1&0&2&3&2\\0&3&0&-3&3\\0&0&0&3&6\end{array}\right] \to \left[\begin{array}{cccc|c}1&0&2&3&2\\0&1&0&-1&1\\0&0&0&1&2\end{array}\right] = [U \mid c]$$

$$\to \left[\begin{array}{cccc|c}1&0&2&0&-4\\0&1&0&0&3\\0&0&0&1&2\end{array}\right] = [R \mid d] \qquad \begin{cases} x_1 + 2x_3 = -4 \\ x_2 = 3 \\ x_4 = 2 \end{cases}$$

$$\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix} + \begin{bmatrix}-4\\3\\0\\2\end{bmatrix}, \quad x_p = \begin{bmatrix}-4\\3\\0\\2\end{bmatrix}, \quad x_n = x_3\begin{bmatrix}-2\\0\\1\\0\end{bmatrix} \text{ for } x_3 \in R$$

5.

$$A = \begin{bmatrix}1&2&5&0&5\\0&0&c&2&2\\0&0&0&d&2\end{bmatrix} \to \begin{bmatrix}1&2&5&0&5\\0&0&c&0&2-\frac{4}{d}\\0&0&0&d&2\end{bmatrix}$$, If $c=0$ and $d=2$, the rank $= 2$

$$B = \begin{bmatrix}c&d\\d&c\end{bmatrix} = \begin{bmatrix}c&d\\0&c-\frac{d^2}{c}\end{bmatrix}$$ If $c - \frac{d^2}{c} \neq 0$, $c \neq \pm d$, the rank $= 2$

$\therefore$ If $c=0$ and $d=2$, these rank $= 2$

6.

(a) If the dimension is zero, there is no basis and vector, the vector doesn't exist.

(b) If the dimension is one, every vectors in the space are the one basis times a constant.

Consider the vectors (x_1, x_2, x_3, x_4) and (x_1, x_2, x_4, x_3). The difference $(0, 0, x_3 - x_4, x_4 - x_3)$ must be the product of the basis and a constant.

Consider the 24 vectors. There are difference of two vectors that have a 0 at one axis with difference of two numbers of (x_1, x_2, x_3, x_4) on other axis.

To make every vectors the product of the basis and a constant, $x_1 = x_2 = x_3 = x_4$

8.

(1) It's row 3 - 2× (row 2) + row 1

(2)

$$A^T = \begin{bmatrix}1&4&7\\2&5&8\\3&6&9\end{bmatrix} \to R = \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix} \qquad N(A^T) = \left\{[x] \,\middle|\, [x] = c\begin{bmatrix}1\\-2\\1\end{bmatrix}, c \in \mathbb{R}\right\}$$

$$A = \begin{bmatrix}1&2&3\\4&5&6\\7&8&9\end{bmatrix} \to R = \begin{bmatrix}1&0&-1\\0&1&2\\0&0&0\end{bmatrix} \qquad N(A) = \left\{[x] \,\middle|\, [x] = c\begin{bmatrix}1\\-2\\1\end{bmatrix}, c \in \mathbb{R}\right\}$$

9.

(a)

$$A = \begin{bmatrix}1&0&-1\\-1&1&0\\0&-1&1\end{bmatrix} \to R = \begin{bmatrix}1&0&-1\\0&1&-1\\0&0&0\end{bmatrix} \qquad \therefore X = c\begin{bmatrix}1\\1\\1\end{bmatrix} \text{ for } c \in R \text{ if } AX = 0$$

The nullspace of A is $\left\{[x] \,\middle|\, [x] = c\begin{bmatrix}1\\1\\1\end{bmatrix}, c \in R\right\}$, the dimension $= 1$

(b)

$$A^T = \begin{bmatrix}1&-1&0\\0&1&-1\\-1&0&1\end{bmatrix} \to R^T = \begin{bmatrix}1&-1&0\\0&1&-1\\0&0&0\end{bmatrix}, \quad R = \begin{bmatrix}1&0&0\\-1&1&0\\0&-1&0\end{bmatrix}$$

$C(A) = \left\{[x] \,\middle|\, [x] = a\begin{bmatrix}1\\-1\\0\end{bmatrix} + b\begin{bmatrix}0\\1\\-1\end{bmatrix}, a, b \in \mathbb{R}\right\}$, the dimension $= 2$