EE2030Linear Algebra

Homework#6 Solution

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1. The equations are $\begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = A \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$ with A= $\begin{bmatrix} .5 & .5 \\ 1 & 0 \end{bmatrix}$. The matrix has $\lambda_1 = 1, \lambda_2 = -\frac{1}{2}$ with x_1 = (1,1), x_2 = (1,-2)
(b)$A^n = X\Lambda^{-1}X^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (-.5)^n \end{bmatrix} \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{bmatrix} \rightarrow A^\infty = \begin{bmatrix} \frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}$

2. $R = S\sqrt{\Lambda}S^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & \\ & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} /2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $R^2 = A$.
$\sqrt{B}$ needs $\lambda = \sqrt{9}$ and $\sqrt{-1}$, trace (their sum) is not real so $\sqrt{B}$ cannot be real.
Note that $\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ has *two* imaginary eigenvalues $\sqrt{-1} = i$ and -i, real trace 0, real square root $\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

3. $d(v+w)/dt = (w\text{-}v)+(v\text{-}w) = 0$, so the total $v+w$ is constant.
$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$ has $\lambda_1 = 0$ and $\lambda_2 = -2$ with $x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
$\begin{bmatrix} v(0) \\ w(0) \end{bmatrix} = \begin{bmatrix} 30 \\ 10 \end{bmatrix} = 20 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 10 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ leads to
$v(1) = 20 + 10e^{-2} \quad v(\infty) = 20, w(1) = 20 - 10e^{-2} \quad w(\infty) = 20$

4. $A = \begin{bmatrix} 1 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \mathbf{1} & 0 \\ 0 & \mathbf{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & \frac{1}{2} \end{bmatrix}$. Then $e^{At} = \begin{bmatrix} \mathbf{e^t} & \frac{1}{2}(e^{3t} - e^t) \\ 0 & \mathbf{e^{3t}} \end{bmatrix}$.
At $t = 0$, $e^{At} = I$ and $\Lambda e^{At} = A$.

5. M is skew-symmetric and **orthogonal**; $\lambda's$ must be i, i, -i, -i to have trace zero.

6. A is invertible, orthogonal, permutation, diagonalizable, Markov; B is projection, diagonalizable, Markov. A allows QR, $S\Lambda S^{-1}$, $Q\Lambda Q^T$; B allows $S\Lambda S^{-1}$ and $Q\Lambda Q^T$.

7. A= $\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$ has pivots 2, $\frac{3}{2}$, $\frac{4}{3}$;
$B = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$ is singular;$B \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

8. A is positive definite when $s > 8$; T is positive definite when $t > 5$ by determinants.

9. Eight families of similar matrices; six matrices have $\lambda = 0, 1$(one family); three matrices have $\lambda = 1, 1$ and three have $\lambda = 0, 0$(two families each!); one has $\lambda = 1, -1$; one has $\lambda = 2, 0$;
two matrices have $\lambda = \frac{1}{2}(1 \pm \sqrt{5})$ (they are in one family).

10. Let $M = (m_{ij})$. Then

$$JM = \begin{bmatrix} m_{21} & m_{22} & m_{23} & m_{24} \\ 0 & 0 & 0 & 0 \\ m_{41} & m_{42} & m_{43} & m_{44} \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \textit{and } MK = \begin{bmatrix} 0 & m_{11} & m_{12} & 0 \\ 0 & m_{21} & m_{22} & 0 \\ 0 & m_{31} & m_{32} & 0 \\ 0 & m_{41} & m_{42} & 0 \end{bmatrix}$$

If JM=MK then

$$m_{21} = m_{22} = m_{24} = m_{41} = m_{42} = m_{44} = 0,$$

which in particular means that the second row is either a multiple of the fourth row, or the fourth row is all 0's. In either of these cases M is not invertible.
Suppose that J were simailar to K. Then there would be some invertible matrix M such that $K = M^{-1}JM$, which would mean that $MK = JM$. But we just showed that in this case M is never invertible! Contradiction. Thus J is not similar to K.

11. $A = U\Sigma V^T = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sigma_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} v_1 & v_2 \end{bmatrix}^{\top} = \dfrac{\begin{bmatrix} 1 & 3 \\ 3 & -1 \end{bmatrix}}{\sqrt{10}} \begin{bmatrix} \sqrt{50} & 0 \\ 0 & 0 \end{bmatrix} \dfrac{\begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}}{\sqrt{5}}$

12. $AA^T = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $u_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$ and $\sigma_2^2 = 1$ with $u_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$.
$AA^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ has $\sigma_1^2 = 3$ with $v_1 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$, $\sigma_2^2 = 1$ with $v_2 = \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ -1/\sqrt{2} \end{bmatrix}$
and $v_3 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$.
Then $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 & v_2 & v_3 \end{bmatrix} = \begin{bmatrix} u_1 & u_2 \end{bmatrix} \begin{bmatrix} \sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = U\Sigma$.