Linear Algebra, EE 10810EECS205004

Final Exam

(Dated: Fall, 2021)

Total scores: 120%

1. (±30%) **[True or False]** Note that: a Right answer for +3%; but a Wrong answer for -3% (答錯倒扣).

F (1) A ~~*normal*~~ self-adjoint *matrix* is a *symmetric matrix*.

F (2) For two square matrices, $\overline{\overline{A}}$ and $\overline{\overline{B}}$, We have $\mathrm{rank}(\overline{\overline{A}}\,\overline{\overline{B}}) = \mathrm{rank}(\overline{\overline{A}})$ if and only if $\overline{\overline{B}}$ is *non-singular*. $\mathrm{rank}(\overline{\overline{B}}) \geq \mathrm{rank}(\overline{\overline{A}})$

T (3) If $\overline{\overline{A}} \in \mathbf{C}^{n\times n}$ is a Hermitian matrix, then $\overline{\overline{A}} + i\,\overline{\overline{I}}_{n\times n}$ is invertible.

F (4) The matrix $\begin{bmatrix} 0 & i \\ i & 2 \end{bmatrix}$ is *Hermitian*. $\begin{pmatrix} 0 & i \\ i & 2 \end{pmatrix}\begin{pmatrix} 0 & -i \\ -i & 2 \end{pmatrix} = \begin{pmatrix} 1 & 2i \\ -2i & 3 \end{pmatrix}$

T (5) The conjugate transpose of a *unitary matrix* is unitary.

T (6) If $\overline{\overline{A}}$ is *positive definite*, then, $-\overline{\overline{A}}$ is *negative definite*.

F (7) The matrix $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ is *orthogonal*.

T (8) Every linear system has a *least squares solution*.

T (9) Eigenvectors of a linear operator $\hat{T}$ are also *generalized eigenvectors* of $\hat{T}$.

T (10) All projections are *self-adjoint*.

$(\overline{\overline{A}} + i)(\overline{\overline{A}} + i)^{-1}$

$= (\overline{\overline{A}} + i)(\overline{\overline{A}}^{-1} - i)$

$= I + I$

2. (10%) **[Unitarily diagonalizable]**

Fill all $a \in \mathbf{C}$ (complex number field), such that the matrix below is *unitarily diagonalizable*,

$$\overline{\overline{\mathbf{U}}} = \begin{bmatrix} i & 4 \\ a & i \end{bmatrix}. \tag{1}$$

3. (15%) **[Polar Decomposition]**

Find the *polar decomposition* of $\overline{\overline{\mathbf{B}}}$:

$$\overline{\overline{\mathbf{B}}} = \begin{bmatrix} 2 & 2 \\ -1 & 1 \end{bmatrix}. \tag{2}$$

$\overline{\overline{B}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}\begin{pmatrix} \frac{3}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \end{pmatrix}$

$\begin{pmatrix} 0 & \sqrt{5} \\ -\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ -1 & 1 \end{pmatrix}$

$\frac{3}{2} + \frac{1}{2} = 2$

$-\frac{2}{3}$ $\frac{-14}{\sqrt{5}}$ $\frac{14}{\sqrt{45}}$ $\frac{-70-70}{\sqrt{45}} \cdots -2$

4. (15%) [**Spectral Theorem**] $-\frac{4}{9} - \frac{14}{5} + \frac{56}{45} = \frac{-20-126+56}{45}$

$-\frac{4}{3}$ $\frac{7}{\sqrt{5}}$ $\frac{28}{\sqrt{45}}$

(a) (10%) Find an orthogonal matrix $\overline{\overline{\mathbf{P}}}$ that diagonalizes

$$\overline{\overline{\mathbf{S}}} = \begin{bmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{bmatrix}. \tag{3}$$

$-\frac{8}{9} + \frac{7}{5} + \frac{112}{45}$

(b) (5%) Perform the spectral decomposition for the matrix $\overline{\overline{\mathbf{S}}}$.

$-\frac{8}{9} + \frac{7}{5} + \frac{112}{45} = -\frac{40}{45} + \frac{63}{45}$

$\frac{-40+63+112}{45} = \frac{112+23}{45} = \frac{135}{45} = 3$

5. (10%) [**Least Squares Approximation**]

Find the parabola $y = C + Dx + Ex^2$ that comes closest (least squares error) to the data points: $(x, y) = (-2, 0), (-1, 0), (0, 1), (1, 2)$, and $(2, 5)$.

$\frac{\sqrt{18}}{3} \times \frac{1}{\sqrt{2}} = \frac{3}{3} =$

6. (20%) [**SVD and Pseudoinverse**]

$\frac{1}{9}$ $-\frac{2}{9}$ $\frac{2}{9}$ $\quad \frac{2}{3}\sqrt{18} \times \frac{1}{\sqrt{2}}$

Consider the matrix: $\frac{1}{18}$ $\quad -\frac{1}{9} + \frac{4}{9} + \frac{4}{9}$ $\quad = \frac{2}{3} \times 3$

$\frac{1}{18} + \frac{2}{9} + \frac{2}{9} = \frac{1+4+4}{18}$ $-\frac{2}{3}\times$

$$\overline{\overline{\mathbf{A}}}_1 = \begin{bmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{bmatrix}, \tag{4}$$

$\sqrt{18}$ $\quad \frac{1}{\sqrt{36}} = \frac{1}{6}$

(a) (10%) Find the corresponding *Singular Value Decomposition*, i.e.,

$$\overline{\overline{\mathbf{A}}}_1 = \overline{\overline{\mathbf{U}}}\,\overline{\overline{\mathbf{\Sigma}}}\,\overline{\overline{\mathbf{V}}}^* \tag{5}$$

(b) (10%) Based on (a), find the *pseudoinverse* of $\overline{\overline{\mathbf{A}}}_1$.

$$\begin{pmatrix} \frac{\sqrt{18}}{3} & 0 \\ -\frac{2\sqrt{18}}{3} & 0 \\ \frac{2\sqrt{18}}{3} & 0 \end{pmatrix}\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1 \\ -2 & 2 \\ 2 & -2 \end{pmatrix}\begin{pmatrix} \frac{1}{18} & -\frac{1}{9} & \frac{1}{9} \\ -\frac{1}{18} & \frac{1}{9} & -\frac{1}{9} \end{pmatrix} = ($$

7. (20%) [**Jordan Canonical Form**]

Given

$$\overline{\overline{\mathbf{A}}}_2 = \begin{bmatrix} 5 & 4 & 3 \\ -1 & 0 & -3 \\ 1 & -2 & 1 \end{bmatrix}. \tag{6}$$

(a) (10%) Express $\overline{\overline{\mathbf{A}}}_2$ being similar to the matrix $\overline{\overline{\mathbf{J}}}$ with Jordan form, i.e.,

$$\overline{\overline{\mathbf{J}}} = \overline{\overline{\mathbf{M}}}^{-1}\,\overline{\overline{\mathbf{A}}}_2\,\overline{\overline{\mathbf{M}}} \tag{7}$$

(b) (10%) Calculate $\overline{\overline{\mathbf{A}}}_2^k$, $k > 1$.

$k=2$ $\quad 4\times1+1\times4$

$k=3$ $\quad 4(4\times1\times1\times4)+4^2$

$k=4$ $\quad 4(4\times$