EECS205003 Linear Algebra, Fall 2020

Quiz # 12, Solutions

Prob. 1:

Use the modified Gram-Schmidt algorithm. Let $\mathbf{u}_1 = (1, -2, 1, 0), \mathbf{u}_2 = (0, 1, 1, 0), \mathbf{u}_3 = (1, 0, 3, -1)$. An orthogonal basis for Span(S) consists of the following three vectors:

$$
\begin{aligned}
\mathbf{z}_1 &= \mathbf{u}_1 = (1, -2, 1, 0), \\
\mathbf{z}_2 &= \mathbf{u}_2 - \text{proj}_{\mathbf{z}_1}\mathbf{u}_2 = (0, 1, 1, 0) - (\frac{-1}{6})(1, -2, 1, 0) = (\frac{1}{6}, \frac{2}{3}, \frac{7}{6}, 0), \\
\mathbf{z}_3 &= \mathbf{u}_3 - \text{proj}_{\mathbf{z}_1}\mathbf{u}_3 - \text{proj}_{\mathbf{z}_2}\mathbf{u}_3 \\
&= (1, 0, 3, -1) - (\frac{4}{6})(1, -2, 1, 0) - (2)(\frac{1}{6}, \frac{2}{3}, \frac{7}{6}, 0) = (0, 0, 0, -1).
\end{aligned}
$$

Prob. 2:

For the least-square solution, consider the system $\mathbf{A}^T\mathbf{A}\mathbf{x} = \mathbf{A}^T\mathbf{b}$:

$$
\begin{bmatrix} 1 & 0 & -2 & 4 \\ -1 & -2 & 2 & 0 \\ 0 & 1 & 3 & -1 \end{bmatrix}
\begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ -2 & 2 & 3 \\ 4 & 0 & -1 \end{bmatrix} \mathbf{x} =
\begin{bmatrix} 1 & 0 & -2 & 4 \\ -1 & -2 & 2 & 0 \\ 0 & 1 & 3 & -1 \end{bmatrix}
\begin{bmatrix} -2 \\ 0 \\ 3 \\ 0 \end{bmatrix}
$$

$$
\Rightarrow \begin{bmatrix} 21 & -5 & -10 \\ -5 & 9 & 4 \\ -10 & 4 & 11 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -8 \\ 8 \\ 9 \end{bmatrix}
$$

$$
\left[\begin{array}{ccc|c} 21 & -5 & -10 & -8 \\ -5 & 9 & 4 & 8 \\ -10 & 4 & 11 & 9 \end{array}\right] \rightarrow
\left[\begin{array}{ccc|c} 1 & -\frac{9}{5} & -\frac{4}{5} & -\frac{8}{5} \\ 0 & -14 & 3 & -7 \\ 0 & \frac{164}{5} & \frac{34}{5} & \frac{128}{5} \end{array}\right] \rightarrow
\left[\begin{array}{ccc|c} 1 & 0 & -\frac{83}{70} & -\frac{7}{10} \\ 0 & 1 & -\frac{3}{14} & \frac{1}{2} \\ 0 & 0 & \frac{484}{35} & \frac{46}{5} \end{array}\right] \rightarrow
\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{43}{484} \\ 0 & 1 & 0 & \frac{311}{484} \\ 0 & 0 & 1 & \frac{161}{242} \end{array}\right]
$$

The least-square solution is $\begin{bmatrix} \frac{43}{484} & \frac{311}{484} & \frac{161}{242} \end{bmatrix}^T$.

Prob. 3:

No. Since when the third equation is changed, the vector and the hyperplane onto which the vector projects are both changed. Therefore we expect the two system have different least-squares solutions. In fact, the least-squares solutions to these two systems are $(-0.1538, 0.9231)$ and $(0.6526, -0.6898)$ respectively.

Prob. 4:

Denote this m-dimensional inner product as V. Since $m > k$, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\} \subsetneq V$. There exists an $\mathbf{v}$ such that $\mathbf{v} \in V$ but $\mathbf{v} \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$.

Let $\mathbf{v}' = \mathbf{v} - \sum_{i=1}^{k} \langle \mathbf{v}, \mathbf{v}_i \rangle \mathbf{v}_i$. Since $\mathbf{v} \notin \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k\}$, $\mathbf{v}' \neq \mathbf{0}$. And since $\langle \mathbf{v}', \mathbf{v}_i \rangle = 0 \ \ \forall i \in \{1, 2, \cdots, k\}$, $\mathbf{v}'$ is orthogonal to $\mathbf{v}_i$ for all $i \in \{1, 2, \cdots, k\}$. Now normalize $\mathbf{v}'$ and choose

$$\mathbf{v}_{k+1} = \frac{\mathbf{v}'}{\|\mathbf{v}'\|},$$

then $\{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \mathbf{v}_{k+1}\}$ is orthonormal in the space V.

Prob. 5:

Let $\mathbf{p}$ be the orthogonal projection of $\mathbf{b}$ onto the column space of $\mathbf{A}$. Then any least squares solution $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{p}$ and

$$\begin{aligned} \mathbf{b}^T \mathbf{A}\mathbf{x} &= [(\mathbf{b} - \mathbf{p}) + \mathbf{p}]^T \mathbf{A}\mathbf{x} = [(\mathbf{b} - \mathbf{p}) + \mathbf{p}]^T \mathbf{p} \\ &= \underbrace{(\mathbf{b} - \mathbf{p})^T \mathbf{p}}_{=\langle \mathbf{p}, \mathbf{b} - \mathbf{p} \rangle = 0} + \mathbf{p}^T \mathbf{p} \geq 0. \end{aligned}$$

Prob. 6:

(a) Since A is nonnegative definite, $\mathbf{e}_i^H A \mathbf{e}_i = a_{ii} \geq 0$ for all $i \in \{0, 1, \cdots, n-1\}$.

(b) Let λ be an eigenvalue of A and $\mathbf{x}$ be a corresponding eigenvector. Then $\mathbf{x}^H A \mathbf{x} = \lambda \mathbf{x}^H \mathbf{x} \geq 0$. Since $\mathbf{x}$ is nonzero, we have $\mathbf{x}^H \mathbf{x} > 0$ and then $\lambda \geq 0$.

(c) Since the complex matrix $\mathbf{A}$ is nonnegative definite, we have $\mathbf{x}^H \mathbf{A} \mathbf{x} \geq 0 \ \forall \mathbf{x} \in \mathbb{C}^n$. Since $\mathbf{x}^H \mathbf{A} \mathbf{x}$ is real $\forall \mathbf{x} \in \mathbb{C}^n$, we have

$$\mathbf{x}^H \mathbf{A} \mathbf{x} = (\mathbf{x}^H \mathbf{A} \mathbf{x})^H = \mathbf{x}^H \mathbf{A}^H \mathbf{x} = (\mathbf{A} \mathbf{x})^H \mathbf{x}$$

which implies that

$$\langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle = \mathbf{x}^H \mathbf{A} \mathbf{x} = (\mathbf{A}\mathbf{x})^H \mathbf{x} = \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle \ \forall \mathbf{x} \in \mathbb{C}^n.$$

(d)

$$\begin{aligned} \langle \mathbf{A}(\mathbf{x}+\mathbf{y}), \mathbf{x}+\mathbf{y} \rangle &= \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{A}\mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{A}\mathbf{y}, \mathbf{y} \rangle \\ \langle (\mathbf{x}+\mathbf{y}), \mathbf{A}(\mathbf{x}+\mathbf{y}) \rangle &= \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{A}\mathbf{y} \rangle \\ \langle \mathbf{A}(\mathbf{x}+i\mathbf{y}), \mathbf{x}+i\mathbf{y} \rangle &= \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle + i\langle \mathbf{A}\mathbf{y}, \mathbf{x} \rangle - i\langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{A}\mathbf{y}, \mathbf{y} \rangle \\ \langle \mathbf{x}+i\mathbf{y}, \mathbf{A}(\mathbf{x}+i\mathbf{y}) \rangle &= \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle + i\langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle - i\langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{A}\mathbf{y} \rangle \end{aligned}$$

By (c) in above, we have $\langle \mathbf{A}(\mathbf{x}+\mathbf{y}), \mathbf{x}+\mathbf{y} \rangle = \langle \mathbf{x}+\mathbf{y}, \mathbf{A}(\mathbf{x}+\mathbf{y}) \rangle$ and $\langle \mathbf{A}(\mathbf{x}+i\mathbf{y}), \mathbf{x}+i\mathbf{y} \rangle = \langle \mathbf{x}+i\mathbf{y}, \mathbf{A}(\mathbf{x}+i\mathbf{y}) \rangle$ so that

$$\begin{aligned} \langle \mathbf{A}\mathbf{x}, \mathbf{x} \rangle + \langle \mathbf{A}\mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle + \langle \mathbf{A}\mathbf{y}, \mathbf{y} \rangle &= \langle \mathbf{x}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle + \langle \mathbf{y}, \mathbf{A}\mathbf{y} \rangle \\ \Rightarrow \langle \mathbf{A}\mathbf{y}, \mathbf{x} \rangle + \langle \mathbf{A}\mathbf{x}, \mathbf{y} \rangle &= \langle \mathbf{y}, \mathbf{A}\mathbf{x} \rangle + \langle \mathbf{x}, \mathbf{A}\mathbf{y} \rangle \end{aligned}$$

and

$$\langle \mathbf{Ax}, \mathbf{x} \rangle + i\langle \mathbf{Ay}, \mathbf{x} \rangle - i\langle \mathbf{Ax}, \mathbf{y} \rangle + \langle \mathbf{Ay}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{Ax} \rangle + i\langle \mathbf{y}, \mathbf{Ax} \rangle - i\langle \mathbf{x}, \mathbf{Ay} \rangle + \langle \mathbf{y}, \mathbf{Ay} \rangle$$
$$\Rightarrow \langle \mathbf{Ay}, \mathbf{x} \rangle - \langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{Ax} \rangle - \langle \mathbf{x}, \mathbf{Ay} \rangle.$$

By subtracting the two equations, we have

$$\langle \mathbf{Ax}, \mathbf{y} \rangle = \langle \mathbf{x}, \mathbf{Ay} \rangle,$$

i.e., A is Hermitian.

(e) For all $\mathbf{x} \in \mathbb{C}^n$, we have

$$\mathbf{x}^H BB^H \mathbf{x} = (B^H \mathbf{x})^H (B^H \mathbf{x}) \geq 0$$

and

$$\mathbf{x}^H B^H B \mathbf{x} = (B\mathbf{x})^H (B\mathbf{x}) \geq 0$$

so that BB^H and $B^H B$ are nonnegative definite.