EE205003 Linear Algebra, 2020 Fall Semester

Quiz # 11

DATE: Dec. 30th, 2020

1. (10%) Let $S = \{(1+i, 3, -4i), (2-i, 2i, 4+5i)\}$ be a subset of $\mathbb{C}^3$. Please find the orthogonal complement $S^\perp$ of S in $\mathbb{C}^3$.

2. (15%) Please verify that $\mathbf{u} = (1, -1, 2)$ and $\mathbf{v} = (4, 2, -1)$ are orthogonal in $\mathbb{R}^3$. Let $W = \text{Span}(\mathbf{u}, \mathbf{v})$. Please find the orthogonal projection of $\mathbf{x} = (1, -1, 1)$ onto W.

3. (15%) Let $\mathbf{x}$ and $\mathbf{y}$ be two vectors in a complex inner product space. Does $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ imply $\langle \mathbf{x}, \mathbf{y} \rangle = 0$? If yes, please show it. Otherwise, give a counterexample.

4. (15%) Please show that in a complex or real inner product space, if $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 = \langle \mathbf{x}, \mathbf{y} \rangle$, then $\mathbf{x} = \mathbf{y}$.

5. (15%) Let $C[a, b]$ be the vector space of all complex-valued continuous functions defined on the interval $[a, b]$. Consider a function $\langle , \rangle : C[a, b] \times C[a, b] \to \mathbb{C}$ with

$$\langle f, g \rangle \triangleq \int_a^b f(x)\overline{g(x)}w(x)dx \quad \forall\, f, g \in C[a, b],$$

where $w(x)$ is a function in $C[a, b]$. What must be assumed of $w(x)$ in order that the function $<, >$ defines an inner product on $C[a, b]$?

6. (15%) The Cauchy-Schwarz inequality in a complex inner product space V is

$$|\langle \mathbf{x}, \mathbf{y} \rangle| \leq \|\mathbf{x}\| \|\mathbf{y}\| \quad \forall\, \mathbf{x}, \mathbf{y} \in V.$$

Please give a necessary and sufficient condition for the equality to hold.

7. (15%) Define an inner product in $\mathbb{R}^3$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle_o \triangleq 2x_1y_1 + x_2y_2 + 2x_3y_3$$

for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$. Fix $\mathbf{y} = (2, -1, 0)$. What is the largest value that $\langle \mathbf{x}, \mathbf{y} \rangle_o$ can attain when $\mathbf{x}$ is a free vector constrained only by $\langle \mathbf{x}, \mathbf{y} \rangle_o \leq 10$? Please find those $\mathbf{x}$ such that $\langle \mathbf{x}, \mathbf{y} \rangle_o$ attains the largest value.