EECS205003 Linear Algebra, Fall 2020

Quiz # 11, Solutions

Prob. 1:

By the definition, the orthogonal complement $S^{\perp}$ of S in $\mathbb{C}^3$ is the set

$$S^{\perp} = \{\mathbf{x} \mid \mathbf{x} \perp \mathbf{s} \text{ for all } \mathbf{s} \in S\}.$$

Since $S = \{(1+i, 3, -4i), (2-i, 2i, 4+5i)\}$, we have the equation

$$\begin{bmatrix} 1-i & 3 & 4i \\ 2+i & -2i & 4-5i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

where $\mathbf{x} = (x_1, x_2, x_3)$ in $\mathbb{C}^3$.

$$\left[\begin{array}{ccc|c} 1-i & 3 & 4i & 0 \\ 2+i & -2i & 4-5i & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{3}{1-i} & \frac{4i}{1-i} & 0 \\ 1 & \frac{-2i}{2+i} & \frac{4-5i}{2+i} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{3+3i}{2} & \frac{-4+4i}{2} & 0 \\ 1 & \frac{-2-4i}{5} & \frac{3-14i}{5} & 0 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & \frac{3+3i}{2} & -2+2i & 0 \\ 0 & \frac{-19-23i}{10} & \frac{26-48i}{10} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & \frac{3+3i}{2} & -2+2i & 0 \\ 0 & 1 & \frac{26-48i}{-19-23i} & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & \frac{-43-140i}{89} & 0 \\ 0 & 1 & \frac{61+151i}{89} & 0 \end{array}\right]$$

Thus the orthogonal complement $S^{\perp}$ S is

$$\left\{ \mathbf{x} \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 43+140i \\ -61-151i \\ 89 \end{bmatrix} t,\ t \in \mathbb{C} \right\}.$$

Prob. 2:

Since $\langle \mathbf{u}, \mathbf{v} \rangle = 1 \times 4 + (-1) \times 2 + 2 \times (-1) = 4 - 2 - 2 = 0$, $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

By Theorem 7.1.11, the orthogonal projection of $\mathbf{x}$ onto W is

$$\begin{aligned} \mathbf{p} &= \langle \mathbf{x}, \frac{\mathbf{u}}{\|\mathbf{u}\|} \rangle \frac{\mathbf{u}}{\|\mathbf{u}\|} + \langle \mathbf{x}, \frac{\mathbf{v}}{\|\mathbf{v}\|} \rangle \frac{\mathbf{v}}{\|\mathbf{v}\|} \\ &= (1 \times \frac{1}{\sqrt{6}} + (-1) \times \frac{-1}{\sqrt{6}} + 1 \times \frac{2}{\sqrt{6}}) \langle \frac{1}{\sqrt{6}}, \frac{-1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \rangle \\ &\quad + (1 \times \frac{4}{\sqrt{21}} + (-1) \times \frac{2}{\sqrt{21}} + 1 \times \frac{-1}{\sqrt{21}}) \langle \frac{4}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{-1}{\sqrt{21}} \rangle \\ &= (\frac{6}{7}, \frac{-4}{7}, \frac{9}{7}). \end{aligned}$$

Prob. 3:

Since $\|\mathbf{x}+\mathbf{y}\|^2 = \langle \mathbf{x}+\mathbf{y}, \mathbf{x}+\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{x}\rangle + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle + \langle \mathbf{y},\mathbf{y}\rangle = \|\mathbf{x}\|^2 + \langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle + \|\mathbf{y}\|^2$, $\|\mathbf{x}+\mathbf{y}\|^2 = \|\mathbf{x}\|^2 + \|\mathbf{y}\|^2$ if and only if $\langle \mathbf{x},\mathbf{y}\rangle + \langle \mathbf{y},\mathbf{x}\rangle = 0$ if and only if $\langle \mathbf{x},\mathbf{y}\rangle + \overline{\langle \mathbf{x},\mathbf{y}\rangle} = 0$ if and only if $\langle \mathbf{x},\mathbf{y}\rangle$ is a pure imaginary number. For example, $\mathbf{x} = (1,1)$ and $\mathbf{y} = (i,i)$ have $\mathbf{x}+\mathbf{y} = (1+i, 1+i)$ and then

$$\|\mathbf{x}\|^2 + \|\mathbf{y}\|^2 = 2+2 = 4 = \|\mathbf{x}+\mathbf{y}\|^2,$$

but $\langle \mathbf{x},\mathbf{y}\rangle = 1(-i) + 1(-i) = -2i \neq 0$. The answer is No.

Prob. 4:

Since $\|\mathbf{x}\|^2 = \|\mathbf{y}\|^2 = \langle \mathbf{x},\mathbf{y}\rangle$, we have

$$\langle \mathbf{x},\mathbf{x}\rangle = \langle \mathbf{x},\mathbf{y}\rangle \text{ and } \langle \mathbf{y},\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{y}\rangle$$

and then

$$\langle \mathbf{x},\mathbf{x}-\mathbf{y}\rangle = 0 \text{ and } \langle \mathbf{x}-\mathbf{y},\mathbf{y}\rangle = 0$$

so that

$$\langle \mathbf{x},\mathbf{x}-\mathbf{y}\rangle = 0 \text{ and } \langle \mathbf{y},\mathbf{x}-\mathbf{y}\rangle = 0.$$

Taking the difference, we have

$$\langle \mathbf{x}-\mathbf{y},\mathbf{x}-\mathbf{y}\rangle = 0$$

which shows that $\mathbf{x}-\mathbf{y} = \mathbf{0}$ and then $\mathbf{x} = \mathbf{y}$.

Prob. 5:

We check the four axioms of an inner product.

1. It is clear that when $f \equiv 0$,

   $$\langle f, f\rangle = \int_a^b f(x)\overline{f(x)}w(x)dx = \int_a^b 0\ w(x)dx = 0.$$

   To have $\langle f, f\rangle > 0$ whenever $f \neq 0$, i.e.,

   $$\langle f, f\rangle = \int_a^b f(x)\overline{f(x)}w(x)dx = \int_a^b |f(x)|^2 w(x)dx > 0 \quad \forall f \neq 0,$$

   we must have $w(x) \geq 0\ \forall x \in [a,b]$ and $w(x)$ not identical to the zero function in any subinterval $[c,d]$ of $[a,b]$ with $c < d$.

2. $\langle f, g\rangle = \overline{\langle g, f\rangle}$.

   $$\langle f, g\rangle = \int_a^b f(x)\overline{g(x)}w(x)dx = \int_a^b \overline{g(x)\overline{f(x)}w(x)}dx = \overline{\langle g, f\rangle}$$

   since $w(x) \geq 0$ for all $x \in [a,b]$.

3. $\langle f+h,g\rangle = \langle f,g\rangle + \langle h,g\rangle$.

$$\begin{aligned}\langle f+h,g\rangle &= \int_a^b (f(x)+h(x))\overline{g(x)}w(x)dx \\ &= \int_a^b f(x)\overline{g(x)}w(x)dx + \int_a^b h(x)\overline{g(x)}w(x)dx\end{aligned}$$

4. $\langle \alpha f,g\rangle = \alpha\langle f,g\rangle$

$$\langle \alpha f,g\rangle = \int_a^b \alpha f(x)\overline{g(x)}w(x)dx = \alpha\int_a^b f(x)\overline{g(x)}w(x)dx = \alpha\langle f,g\rangle.$$

We conclude that $w(x)$ should be non-negative on $[a,b]$ and cannot be identical to the zero function in any subinterval $[c,d]$ of $[a,b]$ with $c<d$.

Prob. 6:

We redo the proof of the Cauchy-Schwarz inequality in a complex inner product space V.

If $\mathbf{y}=\mathbf{0}$, then $|\langle \mathbf{x},\mathbf{y}\rangle| = 0 = \|\mathbf{x}\|\|\mathbf{y}\|$ and the equality of the Cauchy-Schwarz inequality holds. In this case, $\{\mathbf{x},\mathbf{y}\}$ is a linearly dependent set.

Assume $\mathbf{y}\neq\mathbf{0}$. Let

$$\alpha = \begin{cases} \dfrac{|\langle \mathbf{y},\mathbf{x}\rangle|}{\langle \mathbf{y},\mathbf{x}\rangle}, & \text{if } \langle \mathbf{y},\mathbf{x}\rangle \neq 0, \\ 1, & \text{if } \langle \mathbf{y},\mathbf{x}\rangle = 0. \end{cases}$$

It is clear that $|\alpha|=1$ and $\alpha\langle \mathbf{y},\mathbf{x}\rangle = |\langle \mathbf{y},\mathbf{x}\rangle|$. Then for all real number r, we have

$$\begin{aligned}0 &\le \langle \mathbf{x}-\gamma\alpha\mathbf{y}, \mathbf{x}-\gamma\alpha\mathbf{y}\rangle = \langle \mathbf{x},\mathbf{x}\rangle - \gamma\alpha\langle \mathbf{y},\mathbf{x}\rangle - \gamma\bar{\alpha}\langle \mathbf{x},\mathbf{y}\rangle + \gamma^2|\alpha|^2\langle \mathbf{y},\mathbf{y}\rangle \\ &= \langle \mathbf{x},\mathbf{x}\rangle - 2\gamma|\langle \mathbf{y},\mathbf{x}\rangle| + \gamma^2\langle \mathbf{y},\mathbf{y}\rangle \\ &= \langle \mathbf{y},\mathbf{y}\rangle\left(\gamma - \frac{|\langle \mathbf{y},\mathbf{x}\rangle|}{\langle \mathbf{y},\mathbf{y}\rangle}\right)^2 + \left(\langle \mathbf{x},\mathbf{x}\rangle - \frac{|\langle \mathbf{y},\mathbf{x}\rangle|^2}{\langle \mathbf{y},\mathbf{y}\rangle}\right).\end{aligned}$$

By letting $\gamma = \frac{|\langle \mathbf{y},\mathbf{x}\rangle|}{\langle \mathbf{y},\mathbf{y}\rangle}$, we have

$$0 \le \langle \mathbf{x},\mathbf{x}\rangle - \frac{|\langle \mathbf{y},\mathbf{x}\rangle|^2}{\langle \mathbf{y},\mathbf{y}\rangle}.$$

And the equality holds if and only if $\mathbf{x}-\gamma\alpha\mathbf{y}=\mathbf{0}$, i.e., $\mathbf{x} = \frac{\alpha|\langle \mathbf{y},\mathbf{x}\rangle|}{\langle \mathbf{y},\mathbf{y}\rangle}\mathbf{y}$.

Thus if the equality of the Cauchy-Schwarz inequality holds, then $\{\mathbf{x},\mathbf{y}\}$ is a linearly dependent set. Conversely, assume that $\{\mathbf{x},\mathbf{y}\}$ is a linear dependent set.

Then either $\mathbf{x} = \beta\mathbf{y}$ or $\mathbf{y} = \gamma\mathbf{x}$ for some complex numbers $\beta,\ \gamma$. In the former case, we have

$$|\langle \mathbf{x}, \mathbf{y} \rangle| = |\beta|\langle \mathbf{y}, \mathbf{y} \rangle = |\beta|\|\mathbf{y}\|^2 = \|\beta\mathbf{y}\|\|\mathbf{y}\| = \|\mathbf{x}\|\|\mathbf{y}\|.$$

And in the latter case,

$$|\langle \mathbf{x}, \mathbf{y} \rangle| = |\gamma|\langle \mathbf{x}, \mathbf{x} \rangle = |\gamma|\|\mathbf{x}\|^2 = \|\mathbf{x}\|\|\gamma\mathbf{x}\| = \|\mathbf{x}\|\|\mathbf{y}\|.$$

In both cases, the equality of the Cauchy-Schwarz inequality holds.

We conclude that the equality of the Cauchy-Schwarz inequality holds if and only if $\{\mathbf{x}, \mathbf{y}\}$ is a linealy dependent set.

<u>**Prob. 7:**</u>

By the Cauchy-Schwarz inequality in an inner product space, we have

$$|\langle \mathbf{x}, \mathbf{y} \rangle_o| \le \|\mathbf{x}\|_o\|\mathbf{y}\|_o$$

with equality holds if and only if $\{\mathbf{x}, \mathbf{y}\}$ is a linear dependent set. Since $\mathbf{y} = (2, -1, 0)$,

$$\|\mathbf{y}\|_o = \sqrt{2 \times 2^2 + 1 \times (-1)^2 + 2 \times 0^2} = 3.$$

Since $\langle \mathbf{x}, \mathbf{x} \rangle_o \le 10$, we have

$$|\langle \mathbf{x}, \mathbf{y} \rangle_o| \le 3 \times \sqrt{10} = 3\sqrt{10}.$$

The largest value that $\langle \mathbf{x}, \mathbf{y} \rangle_o$ can attain is $3\sqrt{10}$. And the largest value $3\sqrt{10}$ is attained if and only if $\mathbf{x} = \alpha\mathbf{y}$ for some $\alpha > 0$, where

$$\langle \alpha\mathbf{y}, \mathbf{y} \rangle_o = \alpha\|\mathbf{y}\|_o^2 = 9\alpha = 3\sqrt{10}.$$

Thus we have $\alpha = \frac{\sqrt{10}}{3}$ and then $\mathbf{x} = (\frac{\sqrt{10}}{3})(2, -1, 0)$ attains the largest value $\langle \mathbf{x}, \mathbf{y} \rangle_o = 3\sqrt{10}$.