# EECS205003 Linear Algebra, Fall 2020

## Quiz # 10, Solutions

**Prob. 1:**

No. From the statement, the number of columns in $A$ is $n = 6$ and the dimension of Ker($A$) is 1 so that Rank($A$) $= n-$ Nullity($A$) $= 6 - 1 = 5$ by the **Rank-Nullity Theorem**. Since the column rank of $A$ is equal to the rank of $A$ and is less than 7 $=$ Dim($\mathbb{R}^7$), the column space of $A$ is a proper subspace of $\mathbb{R}^7$, which implies that the equation $A\mathbf{x} = \mathbf{b}$ is not solvable for all $\mathbf{b}$.

**Prob. 2:**

The linear system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$ if and only if $\text{Col}(A) = \mathbb{R}^m$ if and only if $\text{Dim}(\text{Col}(A)) = m$ if and only if $\text{Rank}(A) = m$ if and only if $\text{Rank}(A^T) = m$ if and only if $\text{Nullity}(A^T) = m - \text{Rank}(A^T) = 0$ if and only if $\text{Ker}(A^T) = \{0\}$.

**Prob. 3:** Let $\mathbf{u}_1 = (2, -2)$ and $\mathbf{u}_2 = (1, 5)$. The transition matrix for changing $B$-coordinates to $C$-coordinates is $P = [[\mathbf{e}_1]_C [\mathbf{e}_2]_C]$, where

$$[\mathbf{u}_1\ \mathbf{u}_2][\mathbf{e}_1]_C = \mathbf{e}_1 \text{ and } [\mathbf{u}_1\ \mathbf{u}_2][\mathbf{e}_2]_C = \mathbf{e}_2.$$

Thus to find the transition matrix $P$, we apply elementary row operations on the following augmented matrix:

$$\left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ -2 & 5 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & 1 & 1 & 0 \\ 0 & 6 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 2 & 0 & \frac{5}{6} & \frac{-1}{6} \\ 0 & 6 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{5}{12} & \frac{-1}{12} \\ 0 & 1 & \frac{1}{6} & \frac{1}{6} \end{array}\right]$$

Thus the transition matrix $P$ is $\begin{bmatrix} \frac{5}{12} & \frac{-1}{12} \\ \frac{1}{6} & \frac{1}{6} \end{bmatrix}$.

**Prob. 4:**

Let $\mathbf{x} \in V$. Then $[\mathbf{x}]_C = P[\mathbf{x}]_B$, $[\mathbf{x}]_D = Q[\mathbf{x}]_C$. The $[\mathbf{x}]_D = Q[\mathbf{x}]_C = QP[\mathbf{x}]_B$. Then QP is the transition matrix from $B$-coordinates to $D$-Coordinates.

**Prob. 5:**

Since

$$\begin{aligned} T((1,-1)) &= A\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -3 \\ 4 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y_1\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \\ T((2,3)) &= A\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix} = x_2\begin{bmatrix} 1 \\ -1 \end{bmatrix} + y_2\begin{bmatrix} 2 \\ 3 \end{bmatrix}, \end{aligned}$$

we have a linear system

$$\begin{bmatrix} 1 & 2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} -3 & -1 \\ 4 & 3 \end{bmatrix}.$$

To solve $x_1, y_1$ and $x_2, y_2$, we apply elementary row operations on the augmented matrix

$$\left[\begin{array}{cc|cc} 1 & 2 & -3 & -1 \\ -1 & 3 & 4 & 3 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 2 & -3 & -1 \\ 0 & 5 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{-17}{5} & \frac{-9}{5} \\ 0 & 5 & 1 & 2 \end{array}\right] \to \left[\begin{array}{cc|cc} 1 & 0 & \frac{-17}{5} & \frac{-9}{5} \\ 0 & 1 & \frac{1}{5} & \frac{2}{5} \end{array}\right].$$

Thus we have

$$[T((1,-1))]_B = \begin{bmatrix} \frac{-17}{5} \\ \frac{1}{5} \end{bmatrix} \quad \text{and} \quad [T((2,3))]_B = \begin{bmatrix} \frac{-9}{5} \\ \frac{2}{5} \end{bmatrix}$$

and the matrix representation of $T$ relative to the basis $B$ is

$$[T]_B = [[T((1,-1))]_B [T((2,3))]_B] = \begin{bmatrix} \frac{-17}{5} & \frac{-9}{5} \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}.$$

**Prob. 6:**

Let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Then $A \simeq B$ if and only if there is an invertible matrix $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ such that

$$AQ = QB, \text{ i.e., } \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$$

which is

$$\begin{bmatrix} a_{11}q_{11} + a_{12}q_{21} & a_{11}q_{12} + a_{12}q_{22} \\ a_{21}q_{11} + a_{22}q_{21} & a_{21}q_{12} + a_{22}q_{22} \end{bmatrix} = \begin{bmatrix} b_{11}q_{11} + b_{21}q_{12} & b_{12}q_{11} + b_{22}q_{12} \\ b_{11}q_{21} + b_{21}q_{22} & b_{12}q_{21} + b_{22}q_{22} \end{bmatrix}.$$

Now we have a linear system,

$$\begin{bmatrix} a_{11} - b_{11} & -b_{21} & a_{12} & 0 \\ -b_{12} & a_{11} - b_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22} - b_{11} & -b_{21} \\ 0 & a_{21} & -b_{12} & a_{22} - b_{22} \end{bmatrix} \begin{bmatrix} q_{11} \\ q_{12} \\ q_{21} \\ q_{22} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Let the $4 \times 4$ coefficient matrix be $M$. Then $A \simeq B$ if and only if there is an invertible matrix $Q = \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}$ whose corresponding vector $(q_{11}, q_{12}, q_{21}, q_{22})$ is in $\mathrm{Ker}(M)$.

**Prob. 7:**

Let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ be an arbitrary $2 \times 2$ matrix. Then we have $A^T = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix}$. By applying the necessary and sufficient condition found in the previous problem for checking the similarity of $A$ and $A^T$, the corresponding $4 \times 4$ matrix $M$ is

$$M = \begin{bmatrix} 0 & -a_{12} & a_{12} & 0 \\ -a_{21} & a_{11} - a_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22} - a_{11} & -a_{12} \\ 0 & a_{21} & -a_{21} & 0 \end{bmatrix}.$$

To find $\mathrm{Ker}(M)$, we apply elementary row operations on $M$,

$$\begin{bmatrix} 0 & -a_{12} & a_{12} & 0 \\ -a_{21} & a_{11} - a_{22} & 0 & a_{12} \\ a_{21} & 0 & a_{22} - a_{11} & -a_{12} \\ 0 & a_{21} & -a_{21} & 0 \end{bmatrix} \sim \begin{bmatrix} a_{21} & 0 & a_{22} - a_{11} & -a_{12} \\ 0 & a_{21} & -a_{21} & 0 \\ 0 & a_{11} - a_{22} & a_{22} - a_{11} & 0 \\ 0 & -a_{12} & a_{12} & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & \frac{a_{22}-a_{11}}{a_{21}} & -\frac{a_{12}}{a_{21}} \\ 0 & 1 & -1 & 0 \\ 0 & a_{11}-a_{22} & a_{22}-a_{11} & 0 \\ 0 & -a_{12} & a_{12} & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{a_{22}-a_{11}}{a_{21}} & -\frac{a_{12}}{a_{21}} \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

when $a_{21} \neq 0$. In this case, we have

$$\mathrm{Ker}(M) = \left\{ \left( \frac{a_{11}-a_{22}}{a_{21}}\alpha + \frac{a_{12}}{a_{21}}\beta, \alpha, \alpha, \beta \right) \mid \alpha, \beta \in \mathbb{R} \right\}.$$

If $a_{12} \neq 0$, then with $\alpha = 0, \beta = 1$, we have an invertible matrix $Q = \begin{bmatrix} \frac{a_{12}}{a_{21}} & 0 \\ 0 & 1 \end{bmatrix}$ such that

$$QA^TQ^{-1} = \begin{bmatrix} \frac{a_{12}}{a_{21}} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \frac{a_{21}}{a_{12}} & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A.$$

If $a_{12} = 0$, then with $\alpha = 1, \beta = 0$, we have an invertible matrix $Q = \begin{bmatrix} \frac{a_{11}-a_{22}}{a_{21}} & 1 \\ 1 & 0 \end{bmatrix}$ such that

$$QA^TQ^{-1} = \begin{bmatrix} \frac{a_{11}-a_{22}}{a_{21}} & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -\frac{a_{11}-a_{22}}{a_{21}} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A.$$

When $a_{21} = 0$ but $a_{12} \neq 0$, we have

$$M = \begin{bmatrix} 0 & -a_{12} & a_{12} & 0 \\ 0 & a_{11}-a_{22} & 0 & a_{12} \\ 0 & 0 & a_{22}-a_{11} & -a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & a_{11}-a_{22} & 0 & a_{12} \\ 0 & 0 & a_{22}-a_{11} & -a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & a_{11}-a_{22} & a_{12} \\ 0 & 0 & a_{22}-a_{11} & -a_{12} \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 0 & 0 & a_{11}-a_{22} & a_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case, we have

$$\mathrm{Ker}(M) = \left\{ \left( \alpha, \beta, \beta, \frac{a_{22}-a_{11}}{a_{12}}\beta \right) \mid \alpha, \beta \in \mathbb{R} \right\}.$$

And with $\alpha = 0, \beta = 1$, we have an invertible matrix $Q = \begin{bmatrix} 0 & 1 \\ 1 & \frac{a_{22}-a_{11}}{a_{12}} \end{bmatrix}$ such that

$$QA^TQ^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & \frac{a_{22}-a_{11}}{a_{12}} \end{bmatrix} \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} \frac{a_{11}-a_{22}}{a_{12}} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = A.$$

When $a_{21} = a_{12} = 0$, the matrix $A$ is diagonal so that $A^T = A$ and $A \simeq A^T$. We conclude that the answer to this question is yes, i.e., every $2 \times 2$ real matrix is similar to its transpose.