EECS205003 Linear Algebra, Fall 2020

Quiz # 9, Solutions

Prob. 1:

$$\begin{bmatrix} 3 & 0 & -2 & 4 & 1 \\ 4 & -2 & -5 & 2 & 1 \\ -1 & 1 & -2 & 0 & 1 \\ 4 & -3 & -10 & 0 & 2 \\ 2 & -3 & -1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -2 & 0 & 1 \\ 3 & 0 & -2 & 4 & 1 \\ 4 & -2 & -5 & 2 & 1 \\ 4 & -3 & -10 & 0 & 2 \\ 2 & -3 & -1 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -2 & 0 & 1 \\ 0 & 3 & -8 & 4 & 4 \\ 0 & 2 & -13 & 2 & 5 \\ 0 & 1 & -18 & 0 & 6 \\ 0 & -1 & -5 & -2 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} -1 & 1 & -2 & 0 & 1 \\ 0 & -1 & -5 & -2 & 1 \\ 0 & 3 & -8 & 4 & 4 \\ 0 & 2 & -13 & 2 & 5 \\ 0 & 1 & -18 & 0 & 6 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & -2 & 0 & 1 \\ 0 & -1 & -5 & -2 & 1 \\ 0 & 0 & -23 & -2 & 7 \\ 0 & 0 & -23 & -2 & 7 \\ 0 & 0 & -23 & -2 & 7 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 2 & 0 & -1 \\ 0 & 1 & 5 & 2 & -1 \\ 0 & 0 & 23 & 2 & -7 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Let $B = \{(1, -1, 2, 0, -1), (0, 1, 5, 2, -1), (0, 0, 23, 2, -7)\}$. Since $\text{Span}(S) = \text{Span}(B)$ and B is linearly independent, $\{(1, -1, 2, 0, -1), (0, 1, 5, 2, -1), (0, 0, 23, 2, -7)\}$ is a basis for $\text{Span}(S)$.

Prob. 2:

It is clear that the column space of the following matrix is $\mathbb{R}^3$:

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix}.$$

By elementary row operations, we have

$$\begin{bmatrix} -1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -3 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & -3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 1 \\ 0 & -3 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 1 \\ 0 & 0 & -3 & -2 \end{bmatrix}.$$

We can see that the pivots are in the first, the second and the third columns. Thus by Theorem 5.2.13, $\{(-1, 2, -3), (1, 0, 0), (0, 1, 0)\}$ is a basis for $\mathbb{R}^3$ extended from S.

Prob. 3:

For a $p = a_5t^5 + a_4t^4 + a_3t^3 + a_2t^2 + a_1t + a_0 \in \mathbb{P}_5$, we have $p' = 5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1$ and $p'' = 20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2$ so that

$$\begin{aligned} L(p) &= p'' - p' = (20a_5t^3 + 12a_4t^2 + 6a_3t + 2a_2) - (5a_5t^4 + 4a_4t^3 + 3a_3t^2 + 2a_2t + a_1) \\ &= -5a_5t^4 + (20a_5 - 4a_4)t^3 + (12a_4 - 3a_3)t^2 + (6a_3 - 2a_2)t + (2a_2 - a_1). \end{aligned}$$

Now $p \in \text{Ker}(L)$ if and only if $-5a_5 = 20a_5 - 4a_4 = 12a_4 - 3a_3 = 6a_3 - 2a_2 = 2a_2 - a_1 = 0$ if and only if $a_5 = a_4 = a_3 = a_2 = a_1 = 0$. Thus we have $\text{Ker}(L) = \{a_0 \mid a_0 \in \mathbb{R}\} = \mathbb{P}_0 =$ the set of all constant polynomials.

It is clear that Range(L) is a subset of $\mathbb{P}_4$. We claim that Range(L) is exactly $\mathbb{P}_4$. For a polynomial $q = b_4t^4 + b_3t^3 + b_2t^2 + b_1t + b_9$ in $\mathbb{P}_4$. To find a $p \in \mathbb{P}_5$ such that $L(p) = q$, we have to solve the following system of linear equations

$$\begin{aligned} -5a_5 &= b_4 \\ 20a_5 - 4a_4 &= b_3 \\ 12a_4 - 3a_3 &= b_2 \\ 6a_3 - 2a_2 &= b_1 \\ 2a_2 - a_1 &= b_0 \end{aligned}$$

which has a solution $a_5 = -\frac{b_4}{5}$, $a_4 = -\frac{b_3}{4} - b_4$, $a_3 = -\frac{b_2}{3} - b_3 - 4b_4$, $a_2 = -\frac{b_1}{2} - b_2 - 3b_3 - 12b_4$, $a_1 = -b_0 - b_1 - 2b_2 - 6b_3 - 24b_4$, and a_0 arbitrary. Thus Range(L) $= \mathbb{P}_4$.

Since Codomain(L) $= \mathbb{P}_5$, Dim(Codomain(L)) $=$ Dim($\mathbb{P}_5$) $= 6$.

Prob. 4:

(1) (i) Since $(0, 0, \cdots) \in U$, U is nonempty.

(ii) For all $a = (a_1, a_2, a_3, \cdots), b = (b_1, b_2, b_3, \cdots) \in U$, let $c = (c_1, c_2, c_3, \cdots) = a + b$. Then for all $n \geq 4$, we have

$$\begin{aligned} c_n =& a_n + b_n = (a_{n-1} - 2a_{n-2} - a_{n-3}) + (b_{n-1} - 2b_{n-2} - b_{n-3}) \\ =& (a_{n-1} + b_{n-1}) - 2(a_{n-2} + b_{n-2}) - (a_{n-3} + b_{n-3}) \\ =& c_{n-1} - 2c_{n-2} - c_{n-3}. \end{aligned}$$

Thus $c \in U$ and U is closed under the vector addition.

(iii) For all $x = (x_1, x_2, x_3, \cdots) \in U$ and $\alpha \in \mathbb{R}$, let $y = \alpha x$. Then for all $n \geq 4$, we have

$$y_n = \alpha x_n = \alpha(x_{n-1} - 2x_{n-2} - x_{n-3}) = y_{n-1} - 2y_{n-2} - y_{n-3}.$$

Thus $y \in U$ and U is closed under the scalar multiplication.

Therefore, U is a subspace of $\mathbb{R}^\infty$ by Theorem 5.1.1.

(2) First we note that since every component of a sequence in U is uniquely determined by its first three components, two sequences in U are equal to each other if only if their first three components are the same.

Consider three sequences

$$a = (1, 0, 0, -1, \cdots), b = (0, 1, 0, -2, \cdots) \text{ and } c = (0, 0, 1, 1, \cdots),$$

in U and let $S = \{a, b, c\}$. Since $\alpha_1 a + \alpha_2 b + \alpha_3 c = 0$ if and only if $(\alpha_1, \alpha_2, \alpha_3) = (0, 0, 0)$, S is a linearly independent set.

Since $a, b, c \in U$ and U is a vector space, so we have $\text{Span}(S) \subseteq U$.

For all $x = (x_1, x_2, x_3, \cdots) \in U$, since x has the same first three components with $x_1a + x_2b + x_3c$, we have $x = x_1a + x_2b + x_3b \in \text{Span}(S)$, so $U \subseteq \text{Span}(S)$ and then $U = \text{Span}(S)$. Thus S is a basis for U. Since there are three sequences in S, $\text{Dim}(U) = 3$.

Prob. 5:

(i) Since V contains the zero matrix O, it is a nonempty set.

(ii) For all $A_1, A_2 \in V$, $(A_1 + A_2)B = A_1B + A_2B = O + O = O$, so $A_1 + A_2 \in V$ and V is closed under the vector addition.

(iii) For all $A \in V$ and $\alpha \in \mathbb{R}$, $(\alpha A)B = \alpha(AB) = \alpha O = O$, so V is closed under the scalar multiplication.

From (i), (ii), and (iii), V is a subspace of $M^{n\times n}$ by Theorem 5.1.1.

Since $AB = O$, $B^TA^T = O$. Let $\mathbf{r}_i$ be the ith row of A. Then $B^TA^T = O$ if and only if $\mathbf{r}_i^T \in \text{Ker}(B^T)$ for all $1 \le i \le n$. Thus A is the set of all $n \times n$ matrices such that each row of A is the transpose of a column vector in $\text{Ker}(B^T)$. Let $k = \text{Dim}(\text{Ker}(B^T))$ and $\{\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k\}$ be a basis of $\text{Ker}(B^T)$. Then $A \in V$ if and only if

$$A = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{r}_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^k \alpha_{1j}\mathbf{u}_j^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \sum_{j=1}^k \alpha_{2j}\mathbf{u}_j^T \\ \vdots \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \sum_{j=1}^k \alpha_{nj}\mathbf{u}_j^T \end{bmatrix}$$

$$= \sum_{j=1}^k \alpha_{1j} \begin{bmatrix} \mathbf{u}_j^T \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix} + \sum_{j=1}^k \alpha_{2j} \begin{bmatrix} \mathbf{0} \\ \mathbf{u}_j^T \\ \vdots \\ \mathbf{0} \end{bmatrix} + \sum_{j=1}^k \alpha_{nj} \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \vdots \\ \mathbf{u}_j^T \end{bmatrix} = \sum_{i=1}^n \sum_{j=1}^k \alpha_{ij} M_{ij},$$

where α_{ij} are arbitrary scalars and M_{ij} is the $n\times n$ matrix such that the ith row of M_{ij} is $\mathbf{u}_j^T$ while all the other rows are zero rows. Then $S = \{M_{ij} | 1 \le i \le n, 1 \le j \le k\}$ spans V. Consider a linear relation on S,

$$\sum_{i=1}^n \sum_{j=1}^k \alpha_{ij} M_{ij} = O$$

where α_{ij} are scalars. Then we have

$$\sum_{j=1}^k \alpha_{ij}\mathbf{u}_j^T = \mathbf{0} \ \forall\ 1 \le i \le n \text{ if and only if } \sum_{j=1}^k \alpha_{ij}\mathbf{u}_j = \mathbf{0} \ \forall\ 1 \le i \le n$$

which imply that $\alpha_{ij} = 0$ for all $1 \leq i \leq n, 1 \leq j \leq k$ since $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ is a linearly independent set. This shows that S is a linearly independent subset of V. We conclude that S is a basis for V. Thus $\text{Dim}(V) = |S| = n \cdot k = n \cdot \text{Dim}(\text{Ker}(B^T))$.

Prob. 6:

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \text{Ker}(A) \cap \text{Col}(A^T)$. Since $\mathbf{x} \in \text{Col}(A^T)$, there exist an $\mathbf{y} \in \mathbb{R}^m$ such that $A^T\mathbf{y} = \mathbf{x}$ and then $\mathbf{y}^T A = \mathbf{x}^T$. And since $\mathbf{x} \in \text{Ker}(A)$, we have $A\mathbf{x} = \mathbf{0}$ so that $\mathbf{x}^T\mathbf{x} = \mathbf{y}^T A\mathbf{x} = 0$, which implies $x_1^2 + x_2^2 + \cdots + x_n^2 = 0$ and then $\mathbf{x} = \mathbf{0}$. Thus $\text{Ker}(A) \cap \text{Col}(A^T) = \{\mathbf{0}\}$.