EECS205003 Linear Algebra, Fall 2020

Quiz # 8, Solutions

Prob. 1:

(a) No. $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ and $\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}$ are both invertible, but $\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}+\begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}$ is not invertible.

(b) Denote $S=\left\{\begin{bmatrix}-a & a-b\\b & a-c\end{bmatrix}\middle|\, a, b \text{ and } c\in\mathbb{R}\right\}$. It is clear that S is a nonempty subset of $\mathbb{R}^{2\times2}$.

(i) $\forall \begin{bmatrix}-a_1 & a_1-b_1\\b_1 & a_1-c_1\end{bmatrix}$ and $\begin{bmatrix}-a_2 & a_2-b_2\\b_2 & a_2-c_2\end{bmatrix}\in S$,

$$\begin{aligned}&\begin{bmatrix}-a_1 & a_1-b_1\\b_1 & a_1-c_1\end{bmatrix}+\begin{bmatrix}-a_2 & a_2-b_2\\b_2 & a_2-c_2\end{bmatrix}\\=&\begin{bmatrix}-(a_1+a_2) & (a_1+a_2)-(b_1+b_2)\\(b_1+b_2) & (a_1+a_2)-(c_1+c_2)\end{bmatrix}\in S.\end{aligned}$$

(ii) $\forall \begin{bmatrix}-a & a-b\\b & a-c\end{bmatrix}\in S$ and $\forall\alpha\in\mathbb{R}$,

$$\alpha\cdot\begin{bmatrix}-a & a-b\\b & a-c\end{bmatrix}=\begin{bmatrix}-\alpha\cdot a & \alpha\cdot a-\alpha\cdot b\\\alpha\cdot b & \alpha\cdot a-\alpha\cdot c\end{bmatrix}\in S.$$

Thus S is closed under the vector addition and the scalar multiplication and then is a vector subspace of $\mathbb{R}^{2\times2}$.

Prob. 2:

(a)

$$\begin{aligned}T(U)&=\{(x_1-x_2-2x_3,-2x_1+x_3)|-x_1+2x_2-x_3=0,x_1,x_2,x_3\in\mathbb{R}\}\\&=\{(x_1-x_2-2x_3,-2x_1+x_3)|-x_1+2x_2=x_3,x_1,x_2,x_3\in\mathbb{R}\}\\&=\{(x_1-x_2-2(-x_1+2x_2),-2x_1+(-x_1+2x_2))|x_1,x_2\in\mathbb{R}\}\\&=\{(3x_1-5x_2,-3x_1+2x_2)|x_1,x_2\in\mathbb{R}\}\\&=\{x_1(3,-3)+x_2(-5,2)|x_1,x_2\in\mathbb{R}\}\\&=\mathbb{R}^2.\end{aligned}$$

(b)

$$
\begin{aligned}
T^{-1}(V) &= \{(x_1, x_2, x_3) \in \mathbb{R}^3 | (x_1 - x_2 - 2x_3) - (-2x_1 + x_3) = 0\} \\
&= \{(x_1, x_2, x_3) \in \mathbb{R}^3 | 3x_1 - 3x_3 = x_2\} \\
&= \{(x_1, 3x_1 - 3x_3, x_3) | x_1, x_3 \in \mathbb{R}\} \\
&= \{s(1,3,0) + t(0,-3,1) | s, t \in \mathbb{R}\}.
\end{aligned}
$$

Prob. 3:

Let

$$
A = \begin{bmatrix} 1 & -3 & -2 & 2 \\ 0 & 1 & 2 & -1 \\ 1 & 0 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3 & -2 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 3 & 1 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 4 & -1 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{5} \\ 0 & 1 & 0 & -\frac{3}{5} \\ 0 & 0 & 1 & -\frac{1}{5} \end{bmatrix}
$$

and

$$
B = \begin{bmatrix} -1 & 0 & -2 & 2 \\ 2 & 1 & 0 & -1 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -4 & 3 \\ 0 & -1 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & -4 & 3 \\ 0 & 0 & -2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -5 \\ 0 & 0 & 1 & -2 \end{bmatrix}.
$$

Thus A and B are not row equivalent, so that A and B do not have the same row space.

Prob. 4:

Of course, U is a nonempty set. Let $x = (x_1, x_2, \cdots)$ and y $= (y_1, y_2, \cdots)$ be two elements in U and $\alpha \in \mathbb{R}$. There are N and M such that $x_i = 0$ for all $i > N$ and $y_j = 0$ for all $j > M$. Let $z = x + y = (x_1 + y_1, x_2 + y_2, \cdots)$. Then for all $i > \max(N, M)$, we have

$$
z_i = x_i + y_i = 0 + 0 = 0.
$$

Thus $z \in U$ and U is closed under the vector addition. Let $w = \alpha x = (\alpha x_1, \alpha x_2, \cdots)$. Then for all $i > N$,

$$
w_i = \alpha x_i = \alpha 0 = 0.
$$

Thus $w \in U$ and U is closed under the scalar multiplication. This proves that U is a subspace of $\mathbb{R}^\infty$.

Prob. 5:

(a) Since the product AB exists, we can assume that A is an $m \times n$ matrix and B is an $n \times k$ matrix. From matrix multiplication, we have

$$
AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix},
$$

where $\mathbf{b}_i$ is the i-th column of matrix B. $\forall \mathbf{u} \in \mathrm{Col}(AB)$, there exist $\alpha_1, \alpha_2, \cdots, \alpha_k$ in $\mathbb{R}$ such that

$$\begin{aligned}
\mathbf{u} &= \alpha_1 A\mathbf{b}_1 + \alpha_2 A\mathbf{b}_2 + \cdots + \alpha_k A\mathbf{b}_k \\
&= \alpha_1 \sum_{i=1}^{n} \mathbf{a}_i b_{i1} + \alpha_2 \sum_{i=1}^{n} \mathbf{a}_i b_{i2} + \cdots + \alpha_k \sum_{i=1}^{n} \mathbf{a}_i b_{ik} \\
&= (\sum_{i=1}^{k} \alpha_i b_{1i})\mathbf{a}_1 + (\sum_{i=1}^{k} \alpha_i b_{2i})\mathbf{a}_2 + \cdots + (\sum_{i=1}^{k} \alpha_i b_{ni})\mathbf{a}_n,
\end{aligned}$$

where $\mathbf{a}_i$ is the i-th column of matrix A and b_{ij} is the (ij)-th entry of matrix B. It is obvious that $\mathbf{u} \in \mathrm{Col}(A)$, we can conclude that $\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$.

(b) $\mathrm{Col}(AB) = \mathrm{Col}(A)$ if and only if $\mathrm{Col}(AB) \subseteq \mathrm{Col}(A)$ and $\mathrm{Col}(A) \subseteq \mathrm{Col}(AB)$. We have shown that the former inclusion relation is always true in part (a). $\mathrm{Col}(A) \subseteq \mathrm{Col}(AB)$ only if every column of A is in $\mathrm{Col}(AB)$. It means that for each column $\mathbf{a}_i$ of A where $1 \le i \le n$, there exists $\alpha_{i1}, \cdots, \alpha_{ik} \in \mathbb{R}$ such that $\alpha_{i1} A\mathbf{b}_1 + \alpha_{i2} A\mathbf{b}_2 + \cdots + \alpha_{ik} A\mathbf{b}_k = \mathbf{a}_i$, i.e.,

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix} \begin{bmatrix} \alpha_{i1} \\ \alpha_{i2} \\ \vdots \\ \alpha_{ik} \end{bmatrix} = \mathbf{a}_i.$$

Therefore if $[AB|\mathbf{a}_1 \ \cdots \ \mathbf{a}_n] = [AB|A]$ is consistent, then $\mathrm{Col}(AB) = \mathrm{Col}(A)$.

Prob. 6:

Let A be a nonempty subset in a vector space V. It has been shown that $\mathrm{Span}(A)$ is a subspace of V. It is clear that $A \subseteq \mathrm{Span}(A)$. Suppose that there is a subspace W of V which contains the set A and is smaller than $\mathrm{Span}(A)$. Then there is an element $\mathbf{b} \in \mathrm{Span}(A)$ but $\mathbf{b} \notin W$. Since $\mathbf{b} \in \mathrm{Span}(A)$, there exist $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k$ in A and $\alpha_1, \alpha_2, \cdots, \alpha_k$ in $\mathbb{R}$ such that

$$\mathbf{b} = \alpha_1 \mathbf{a}_1 + \alpha_2 \mathbf{a}_2 + \cdots + \alpha_k \mathbf{a}_k.$$

Since $A \subseteq W$, $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k$ in W. Also since W is a subspace, any linear combination of $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k$ must be in W by the closure properties of vector spaces. Thus we have $\mathbf{b} \in W$, a contradiction. We conclude that $\mathrm{Span}(A)$ is the smallest subspace containing A.