EECS205003 Linear Algebra, Fall 2020

Quiz # 7, Solutions

Prob. 1:

We first note that Det(A)$=(-1)\times 1\times 3\times(-2)\times 1 = 6$, Det($B$)$= 2\times(-2)\times 1\times 3\times(-1) = 12$, and Det($C$)$=(-2)\times 4\times 1\times(-2)\times(-1) = -16$. Since Det($ABC$)=Det($A$)Det($B$)Det($C$), we have Det($ABC$)$=6\times 12\times(-16) = -1152$.

Prob. 2:

$$\text{Det}\left(\begin{bmatrix} 0 & 3 & -2 & -2 \\ -1 & 1 & 0 & -3 \\ 2 & 3 & -2 & -3 \\ 3 & -1 & 0 & 2 \end{bmatrix}\right) = \text{Det}\left(\begin{bmatrix} 0 & 3 & -2 & -2 \\ -1 & 1 & 0 & -3 \\ 2 & 0 & 0 & -1 \\ 3 & -1 & 0 & 2 \end{bmatrix}\right)$$

$$= (-1)^{1+3}(-2)\text{Det}\left(\begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & -1 \\ 3 & -1 & 2 \end{bmatrix}\right) = (-2)\text{Det}\left(\begin{bmatrix} -1 & 1 & -3 \\ 2 & 0 & -1 \\ 2 & 0 & -1 \end{bmatrix}\right) = 0.$$

Prob. 3:

1. True. By Theorem 2 in Section 4.2, Det(AB) $=$ Det(A)Det(B). Now, let $B = A^{-1}$, Det(AA^{-1}) $=$ Det(I) $= 1 =$ Det(A)Det(A^{-1}) which implies that Det(A^{-1}) $= \frac{1}{\text{Det}(A)} = (\text{Det}(A))^{-1}$. Therefore, Det($A^{-2}$) $=$ Det($A^{-1}A^{-1}$) $=$ Det(A^{-1})Det(A^{-1}) $= (\text{Det}(A))^{-1}(\text{Det}(A))^{-1} = (\text{Det}(A))^{-2}$.

2. True. Det(AB^T) $=$ Det(A)Det(B^T) $=$ Det(A)Det(B) (By Theorem 3 in Section 4.2) $=$ Det(B)Det(A) $=$ Det(BA).

3. False. Consider $n = 2$, $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Then, Det($A$) $-$ Det(B) $= 1 - 1 = 0$ is not equal to Det($A - B$) $=$ Det($\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$) $= 4$.

4. False. If we change one entry a_{ij_0} in an $n\times n$ matrix A to a'_{ij_0} to obtain a new matrix A', by Theorem 1 in Section 4.2, Det(A) $= \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\text{Det}(M^{ij}(A))$ will change to Det(A') $= \sum_{j=1, j\neq j_0}^{n}(-1)^{i+j}a_{ij}\text{Det}(M^{ij}(A)) + (-1)^{i+j_0}a'_{ij_0}\text{Det}(M^{ij_0}(A))$. If Det($M^{ij_0}(A)$) equals to 0, then Det($A$) $=$ Det(A'), i.e., the determinant will not change. For example, consider a 2×2 matrix $A = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. If we change the $(1,2)$-entry 0 to -3 to obtain a matrix $B = \begin{bmatrix} 2 & -3 \\ 0 & -1 \end{bmatrix}$, we have Det($A$) $= -2 =$ Det(B).

Prob. 4:

Consider the cofactor expansion of Det(A) of an $n\times n$ matrix A using row i,

$$\text{Det}(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\text{Det}(M^{ij}).$$

There is a j_0 such that $\text{Det}(M^{ij_0}) \neq 0$, otherwise $\text{Det}(A) = 0$, which is a contradiction to the invertibility of A. If we change the (ij_0)-th entry a_{ij_0} of A to

$$a'_{ij_0} = -\frac{(-1)^{i+j_0}}{\text{Det}(M^{ij_0})} \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j} a_{ij} \text{Det}(M^{ij}),$$

then the new matrix A' has

$$\begin{aligned} \text{Det}(A') &= \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j} a_{ij} \text{Det}(M^{ij}) + (-1)^{i+j_0} a'_{ij} \text{Det}(M^{ij_0}) \\ &= \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j} a_{ij} \text{Det}(M^{ij}) - \sum_{j=1, j\neq j_0}^{n} (-1)^{i+j} a_{ij} \text{Det}(M^{ij}) \\ &= 0. \end{aligned}$$

Thus A' is non-invertible.

Prob. 5:

Recall that the greatest common divisor (gcd) of two nonzero integers can be computed by the Euclidean algorithm, which is just a sequence of replacement operations. For example, to find the gcd of -30 and 21, we apply replacement operations alternatively as follows:

$$\begin{array}{ll} -30 \leftarrow -30 + 21 \times 2 = 12 & \qquad 21 \leftarrow 21 + 12 \times (-1) = 9 \\ 12 \leftarrow 12 + 9 \times (-1) = 3 & \qquad 9 \leftarrow 9 + 3 \times (-3) = 0 \\ 3 & \qquad 0 \end{array}$$

Eventually, the original two numbers -30 and 21 are replaced by their gcd 3 and 0. Consider an $n \times n$ matrix A whose entries a_{ij} , $1 \leq i, j \leq n$, are integers. If the first column of A is the zero column, we do nothing on this column. If the first column of A has more than one nonzero entries, says, $a_{1m} \neq 0$ and $a_{1k} \neq 0$, we can find their gcd by applying a sequence of replacement row operations on the mth row and on the kth row of A alternatively, corresponding to the replacement operations on a_{1m} and a_{1k} alternatively in above. Let A' be the resulted matrix. Then the entries of A' are still integers and $\text{Det}(A') = \text{Det}(A)$. Also one of a'_{1m} and a'_{1k} is the gcd of a_{1m} and a_{1k} and the other is 0. The number of nonzero entries in the first column of A' is one less than the number of nonzero entries in the first column of A. We continue this process to eliminate nonzero entries in the first column to zeros one by one till there is only one nonzero entries in the first column. In this process, all the resulted matrices have integer entries and the same determinant values. By a row exchange, we move the only nonzero entries of the first column to the 1st row and change the sign of the determinant. Now we work on the second column but only consider the entries below the first row. With replacement row operations corresponding to the Euclidean algorithm, we can eliminate nonzero entries below the first row one by one. This procedure will turn the matrix A to an upper triangular matrix B with integer entries and $\text{Det}(A) = \pm\text{Det}(B)$. Since $\text{Det}(B)$ is the product of diagonal entries of B, it is an integer. This proves that $\text{Det}(A)$ is an integer.