EE205003 Linear Algebra, 2020 Fall Semester

Quiz # 6

Date: Oct. 28th, 2020

1. Test each of these sets of real-valued functions defined on $\mathbb{R}$ for linear dependence or linear independence.
   (a) (10%) $u_1(t) = 1$, $u_2(t) = 1 - t + t^2$, $u_3(t) = 2 - t^2$.
   (b) (5%) $u_1(t) = \sin^2 t$, $u_2(t) = \cos^2 t$, $u_3(t) = \sin 2t$.

2. (15%) In the axioms for a vector space, can we dispense with the axiom of the existence of additive inverses (Axiom 4), and still prove that $\mathbf{u} + (-1)\mathbf{u} = \mathbf{0}$? (Hint: Use Theorem 2.4.A: In a vector space, there is one and only one zero vector.)

3. (20%) Let V be the set of all positive real numbers together with new definitions of vector addition and scalar multiplication as follows:

$$u \oplus v \triangleq uv \text{ and } \alpha \odot v \triangleq v^{\alpha},$$

   for all $u, v \in V$ and $\alpha \in \mathbb{R}$. Please show that $(V, \oplus, \odot)$ is a vector space over $\mathbb{R}$. (Hint: Show that both vector addition and scalar multiplication have the closure property and the 7 axioms listed in the class lecture are satisfied.)

4. (15%) Consider $p_1(t) = 1 - t + t^2, p_2(t) = 6 - t + 3t^2, p_3(t) = 2 + 3t - t^2$. Does $\{p_1(t), p_2(t), p_3(t)\}$ span $\mathbb{P}_2$? Please give reason(s), otherwise no credits. (Note that $\mathbb{P}_2$ is the vector space of all polynomials of degree ≤ 2.)

5. (15%) Explain this: If S is a linearly dependent set in a vector space, then there is a vector $\mathbf{v}$ in S such that $\text{Span}(S) = \text{Span}(S \backslash \{\mathbf{v}\})$. If there are two such vectors in S, can we remove them both without changing the span?

6. (20%) Is the following example a vector space over $\mathbb{R}$? $V = \mathbb{R}, x \oplus y = x + y + 1, \alpha \otimes x = \alpha x + \alpha$, where $x, y \in V$ and $\alpha \in \mathbb{R}$. Please give reasons, otherwise no credits.

The following axioms must be fulfilled by a vector space V:

(1) $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u} \; \forall \; \mathbf{u}, \mathbf{v} \in V$. (Commutativity of vector addition)

(2) $(\mathbf{u}+\mathbf{v})+\mathbf{w} = \mathbf{u}+(\mathbf{v}+\mathbf{w}) \; \forall \; \mathbf{u}, \mathbf{v}$ and $\mathbf{w} \in V$. (Associativity of vector addition)

(3) There is an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u} \; \forall \; \mathbf{u} \in V$. (Existence of a additive identity)

(4) For each $\mathbf{u} \in V$, there exists an element $\tilde{\mathbf{u}} \in V$ such that $\tilde{\mathbf{u}}+\mathbf{u} = \mathbf{0}$. (Existence of additive inverses)

(5) $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ and $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u} \; \forall \; \mathbf{u}, \mathbf{v} \in V$ and $\alpha, \beta \in \mathbb{R}$. (Distributive law)

(6) $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u}) \; \forall \; \mathbf{u} \in V$ and $\alpha, \beta \in \mathbb{R}$. (Associativity of scalar multiplication)

(7) $1 \cdot \mathbf{u} = \mathbf{u} \; \forall \; \mathbf{u} \in V$. (Unitary property)