# EECS205003 Linear Algebra, Fall 2020

## Quiz # 6, Solutions

**Prob. 1:**

(a) Consider an arbitrary linear relation in the set $\{u_1(t), u_2(t), u_3(t)\}$,

$$a_1u_1(t) + a_2u_2(t) + a_3u_3(t) = a_1 + a_2(1 - t + t^2) + a_3(2 - t^2) = 0,$$

where $a_1, a_2, a_3$ are real numbers. By evaluating the linear relation at $t = -1, 0, 1$, we have

$$\begin{aligned} a_1 + 3a_2 + a_3 &= 0, \\ a_1 + a_2 + 2a_3 &= 0, \\ a_1 + a_2 + a_3 &= 0. \end{aligned}$$

To solve this homogeneous linear system, we find the reduced row echelon form of the coefficient matrix as follows:

$$\begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 1 \\ 0 & -2 & 1 \\ 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

which shows that $a_1 = a_2 = a_3 = 0$ is the only solution. We conclude that $\{u_1, u_2, u_3\}$ is a linearly independent set.

(b) Consider an arbitrary linear relation in the set $\{u_1(t), u_2(t), u_3(t)\}$,

$$\alpha u_1 + \beta u_2 + \gamma u_3 = \alpha \sin^2 t + \beta \cos^2 t + \gamma \sin 2t = 0,$$

where $\alpha, \beta, \gamma$ are real numbers. By evaluating the linear relation at $t = 0, \pi/4, \pi/2$, we have

$$\begin{aligned} \beta &= 0, \\ \frac{1}{2}\alpha + \frac{1}{2}\beta + \gamma &= 0, \\ \alpha &= 0, \end{aligned}$$

which shows that $\alpha = \beta = \gamma = 0$. We conclude that $\{u_1, u_2, u_3\}$ is a linearly independent set.

**Prob. 2:**

Let $V = \{\mathbf{0}, \mathbf{1}\}$.
Define $\oplus$ as logical OR and $\alpha \otimes x = x \forall x \in V, \alpha \in \mathbb{R}$.

A.1 $x \oplus y = y \oplus x \forall x, y \in V$.

A.2 $x \oplus (y \oplus z) = x \oplus (y \oplus z) \forall x, y$ and $z \in V$.

A.3 $\mathbf{0}$ is the additive identity since $\mathbf{0} \oplus \mathbf{0} = \mathbf{0}$ and $\mathbf{1} \oplus \mathbf{0} = \mathbf{1}$.

A.5 $\alpha \otimes (x \oplus y) = x \oplus y = \alpha \otimes x \oplus \alpha \otimes y$ and $(\alpha+\beta) \otimes x = x = x \oplus x = \alpha \otimes x \oplus \beta \otimes x \forall x, y \in V$ and $\alpha \in \mathbb{R}$.

A.6 $(\alpha\beta) \otimes x = x = \beta \otimes x = \alpha \otimes (\beta \otimes x) \forall x \in V$ and $\alpha, \beta \in \mathbb{R}$.

A.7 $1 \otimes x = x \forall x \in V$.

We can see that $\mathbf{0}$ is the additive identity, but $\mathbf{1}$ doesn't have an additive inverse. Therefore the other 6 axioms don't imply the axiom 4, and $\mathbf{1} \oplus (-1) \otimes \mathbf{1} = \mathbf{1} \neq \mathbf{0}$.

**Prob. 3:**

Let $u, v$ be vectors in $V$ and $\alpha$ a scalar in $\mathbb{R}$. Since $u, v$ are positive real numbers, their product $uv$ is also a positive number. Also for any positive number $u$ and any real number $\alpha$, $u^\alpha$ is a positive number. Thus we have

$$u \oplus v \triangleq uv \in V \text{ and } \alpha \odot v \triangleq v^\alpha \in V$$

and then both the vector addition $\oplus$ and the scalar multiplication $\odot$ have the closure property. We next verify the 7 axioms as follows. Let $u, v, w \in V$ and $\alpha, \beta \in \mathbb{R}$.

1. (Commutativity of vector addition)

$$u \oplus v = uv = vu = v \oplus u.$$

2. (Associativity of vector addition)

$$(u \oplus v) \oplus w = (uv) \oplus w = (uv)w = u(vw) = u \oplus (vw) = u \oplus (v \oplus w).$$

3. (Existence of a zero vector)

$$u \oplus 1 = u\,1 = u,$$

which shows that 1 is a zero vector for the vector addition $\oplus$.

4. (Existence of an additive inverse) $\frac{1}{u}$ is in $V$ such that

$$\frac{1}{u} \oplus u = \frac{1}{u}\, u = 1,$$

which shows that $\frac{1}{u}$ is an additive inverse of $u$.

5. (Distributive laws)

$$\alpha \odot (u \oplus v) = \alpha \odot (uv) = (uv)^\alpha = u^\alpha v^\alpha = u^\alpha \oplus v^\alpha = (\alpha \odot u) \oplus (\alpha \odot v)$$

and

$$(\alpha + \beta) \odot u = u^{(\alpha+\beta)} = u^\alpha u^\beta = u^\alpha \oplus u^\beta = (\alpha \odot u) \oplus (\beta \odot u).$$

6. (Associativity of scalar multiplication)

$$\alpha \odot (\beta \odot u) = \alpha \odot u^\beta = (u^\beta)^\alpha = u^{\alpha\beta} = (\alpha\beta) \odot u.$$

7. (Unitary property)

$$1 \odot u = u^1 = u.$$

**Prob. 4:**

The answer is no.

The set $\{p_1(t), p_2(t), p_3(t)\}$ spans $P_2$ if and only if for any $p(t) = a_0 + a_1 t + a_2 t^2 \in P_2$, there exists $x, y, z \in \mathbb{R}$ such that

$$\begin{aligned} a_0 + a_1 t + a_2 t^2 &= xp_1(t) + yp_2(t) + zp_3(t) = x(1 - t + t^2) + y(6 - t + 3t^2) + z(2 + 3t - t^2) \\ &= (x + 6y + 2z) + (-x - y + 3z)t + (x + 3y - z)t^2, \end{aligned}$$

which is equivalent to solving the linear system

$$\begin{bmatrix} 1 & 6 & 2 \\ -1 & -1 & 3 \\ 1 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

By elementary row operations on the augmented matrix, we have

$$\left[\begin{array}{ccc|c} 1 & 6 & 2 & a_0 \\ -1 & -1 & 3 & a_1 \\ 1 & 3 & -1 & a_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 6 & 2 & a_0 \\ 0 & 5 & 5 & a_0 + a_1 \\ 0 & -3 & -3 & -a_0 + a_2 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 6 & 2 & a_0 \\ 0 & 1 & 1 & \frac{1}{5}a_0 + \frac{1}{5}a_1 \\ 0 & 0 & 0 & -\frac{2}{5}a_0 + \frac{3}{5}a_1 + a_2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -4 & -\frac{1}{5}a_0 - \frac{6}{5}a_2 \\ 0 & 1 & 1 & \frac{1}{5}a_0 + \frac{1}{5}a_1 \\ 0 & 0 & 0 & -\frac{2}{5}a_0 + \frac{3}{5}a_1 + a_2 \end{array}\right]$$

which says that when $-\frac{2}{5}a_0 + \frac{3}{5}a_1 + a_2 \neq 0$, $p(t)$ is not in the $\text{Span}(\{p_1(t), p_2(t), p_3(t)\})$. We conclude that the set $\{p_1(t), p_2(t), p_3(t)\}$ cannot span $P_2$.

**Prob. 5:**

Consider a linearly dependent subset $S$ in a vector space. Then there is a non-trivial linear relation on $S$, i.e.,

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_n\mathbf{v}_n = 0$$

for some distinct vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $S$ and scalars $a_1, a_2, \ldots, a_n$, not all zeros, for some $n \geq 1$. Since scalars $a_1, a_2, \ldots, a_n$ are not all zeros, $a_i \neq 0$ for some $1 \leq i \leq n$. Let $\mathbf{v} = \mathbf{v}_i$ and then

$$\mathbf{v} = -\frac{a_1}{a_i}\mathbf{v}_1 - \cdots - \frac{a_{i-1}}{a_i}\mathbf{v}_{i-1} - \frac{a_{i+1}}{a_i}\mathbf{v}_{i+1} - \cdots - \frac{a_n}{a_i}\mathbf{v}_n.$$

Consider an arbitrary vector $\mathbf{u} \in \text{Span}(S)$,

$$\mathbf{u} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k,$$

where $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are distinct vectors in $S$ and $\alpha_1, \alpha_2, \ldots, \alpha_k$ are scalars with $k \geq 1$. If $\mathbf{u}_i \neq \mathbf{v}$ for all $1 \leq i \leq k$, then $\mathbf{u} \in \text{Span}(S \backslash \{\mathbf{v}\})$. If there is a $\mathbf{u}_j = \mathbf{v}$, then

$$\begin{aligned} \mathbf{u} &= \alpha_1\mathbf{u}_1 + \cdots + \alpha_{j-1}\mathbf{u}_{j-1} + \alpha_j\mathbf{v} + \alpha_{j+1}\mathbf{u}_{j+1} + \cdots + \alpha_k\mathbf{u}_k \\ &= \alpha_1\mathbf{u}_1 + \cdots + \alpha_{j-1}\mathbf{u}_{j-1} + \alpha_{j+1}\mathbf{u}_{j+1} + \cdots + \alpha_k\mathbf{u}_k \\ &\quad -\frac{\alpha_j a_1}{a_i}\mathbf{v}_1 - \cdots - \frac{\alpha_j a_{i-1}}{a_i}\mathbf{v}_{i-1} - \frac{\alpha_j a_{i+1}}{a_i}\mathbf{v}_{i+1} - \cdots - \frac{\alpha_j a_n}{a_i}\mathbf{v}_n \end{aligned}$$

which is in $\mathrm{Span}(S \setminus \{\mathbf{v}\})$ since $\mathbf{u}_1, \ldots, \mathbf{u}_{j-1}, \mathbf{u}_{j+1}, \ldots, \mathbf{u}_k$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \ldots, \mathbf{v}_n$ are vectors in $S \setminus \{\mathbf{v}\}$. Thus we have $\mathrm{Span}(S) \subseteq \mathrm{Span}(S \setminus \{\mathbf{v}\})$. But since $S \setminus \{\mathbf{v}\} \subseteq S$, we have $\mathrm{Span}(S \setminus \{\mathbf{v}\}) \subseteq \mathrm{Span}(S)$. We can conclude that $\mathrm{Span}(S) = \mathrm{Span}(S \setminus \{\mathbf{v}\})$.

If there are two such vectors in $S$, we cannot remove them both without changing the span in general. For example, $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$, where $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. But $\mathrm{Span}(S) = \mathbb{R}^2 \neq \mathrm{Span}\left( S \setminus \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \right) = \mathrm{Span}\left( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \right)$.

**Prob. 6:**

No.

For any vectors $x, y \in V = \mathbb{R}$ and any scalar $\alpha \in \mathbb{R}$, we have

$$x \oplus y \triangleq x + y + 1 \in V = \mathbb{R} \text{ and } \alpha \otimes x \triangleq \alpha x + \alpha \in V = \mathbb{R}$$

since real numbers are closed under multiplication and addition. Thus both the vector addition $\oplus$ and the scalar multiplication $\otimes$ have the closure property. We next verify the 7 axioms as follows.

1. (Commutativity of vector addition)

$$x \oplus y = x + y + 1 = y + x + 1 = y \oplus x.$$

2. (Associativity of vector addition)

$$\begin{aligned} (x \oplus y) \oplus z &= (x + y + 1) \oplus z \\ &= (x + y + 1) + z + 1 \\ &= x + (y + z + 1) + 1 \\ &= x \oplus (y + z + 1) \\ &= x \oplus (y \oplus z). \end{aligned}$$

3. (Existence of a zero vector)

$$x \oplus (-1) = x + (-1) + 1 = x,$$

which shows that (-1) is a zero vector for the vector addition $\oplus$.

4. (Existence of an additive inverse)

$$(-x - 2) \oplus x = (-x - 2) + x + 1 = -1,$$

which shows that $(-x - 2)$ is an additive inverse of $x$.

5. (Distributive laws)

$$\begin{aligned} \alpha \otimes (x \oplus y) &= \alpha \otimes (x + y + 1) \\ &= \alpha(x + y + 1) + \alpha \\ &= \alpha x + \alpha y + 2\alpha \end{aligned}$$

$$\begin{aligned}(\alpha \otimes x) \oplus (\alpha \otimes y) &= (\alpha x + \alpha) \oplus (\alpha y + \alpha) \\ &= \alpha x + \alpha y + 2\alpha + 1 \\ &\neq \alpha \otimes (x \oplus y).\end{aligned}$$

and

$$(\alpha + \beta) \otimes x = (\alpha + \beta)x + (\alpha + \beta)$$

$$\begin{aligned}(\alpha \otimes x) \oplus (\beta \otimes x) &= (\alpha x + \alpha) \oplus (\beta x + \beta) \\ &= \alpha x + \beta x + \alpha + \beta + 1 \\ &\neq (\alpha + \beta) \otimes x.\end{aligned}$$

6. (Associativity of scalar multiplication)

$$\begin{aligned}\alpha \otimes (\beta \otimes x) &= \alpha \otimes (\beta x + \beta) \\ &= \alpha\beta x + \alpha\beta + \alpha\end{aligned}$$

$$\begin{aligned}(\alpha\beta) \otimes x &= \alpha\beta x + \alpha\beta \\ &\neq \alpha \otimes (\beta \otimes x).\end{aligned}$$

7. (Unitary property)

$$1 \otimes x = x + 1 \neq x.$$

Thus while Axioms 1, 2, 3, 4 are fulfilled, but Axioms 5, 6, 7 are not fulfilled.