EECS205003 Linear Algebra, Fall 2020

Quiz # 5, Solutions

Prob. 1:

1. According to the definition of functions, the domain of T is $\mathbb{R}^4$ and the co-domain of T is $\mathbb{R}^3$.

2. By elementary row operations, we have

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & 0 \\ -2 & 1 & -3 & 1 & 0 \\ 0 & -1 & -3 & 3 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & 0 \\ 0 & 1 & 3 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

and the kernel of T is the solution set of

$$\begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & 3 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{cases} x_1 = -3x_3 + 2x_4 \\ x_2 = -3x_3 + 3x_4 \end{cases} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -3 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ 3 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\text{Ker}(T) = \{s(-3,-3,1,0) + t(2,3,0,1) | t, s \in \mathbb{R}\}$.

3. The range $T(\mathbb{R}^4)$ of T is the column space $\text{Col}(A)$ of the matrix A, i.e., $T(\mathbb{R}^4) = \text{Col}(A) = \text{Span}\{(1,-2,0), (-1,1,-1), (0,-3,-3), (1,1,3)\}$.

4. Since $\text{Ker}(T) \neq \{\mathbf{0}\}$, T is not injective by Theorem 5 in Section 2.3 of the textbook.

5. T is surjective if and only if $T(\mathbb{R}^4) = \mathbb{R}^3$, i.e., given any vector $\mathbf{y} = (y_1, y_2, y_3)$ in $\mathbb{R}^3$, there is a vector $\mathbf{x} = (x_1, x_2, x_3, x_4)$ in $\mathbb{R}^4$ such that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{y}$. By elementary row operations on the augmented matrix, we have

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 1 & y_1 \\ -2 & 1 & -3 & 1 & y_2 \\ 0 & -1 & -3 & 3 & y_3 \end{array}\right] \Rightarrow \left[\begin{array}{cccc|c} 1 & 0 & 3 & -2 & -y_1 - y_2 \\ 0 & 1 & 3 & -3 & -2y_1 - y_2 \\ 0 & 0 & 0 & 0 & -2y_1 - y_2 + y_3 \end{array}\right],$$

which shows that $A\mathbf{x} = \mathbf{y}$ is not always consistent for any given $\mathbf{y}$. Thus we conclude that T is not surjective.

Prob. 2:

Solution 1.
By inspection or by solving a linear system with an augmented matrix

$$\left[\begin{array}{cc|c} -1 & 0 & -3 \\ 1 & 1 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} -1 & 0 & -3 \\ 0 & 1 & -2 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right],$$

we have that $(-3,1) = 3(-1,1) - 2(0,1)$. If T is linear, we must have $T(-3,1) = T(3(-1,1) - 2(0,1)) = 3T(-1,1) - 2T(0,1) = (-3,4,-2)$, which is not $(2,-1,0)$, a contradiction. Thus there is no such a linear transformation.

Solution 2.
A more systematic approach is to find a 3×2 matrix A such that

$$T\left(\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix}, \; T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}, \; T\left(\begin{bmatrix} -3 \\ 1 \end{bmatrix}\right) = A\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}.$$

This is equivalent to solving the matrix equation

$$A\begin{bmatrix} -1 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix}\begin{bmatrix} -1 & 0 & -3 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ 2 & 1 & -1 \\ -2 & -2 & 0 \end{bmatrix}.$$

By taking the transpose on both sides, we have

$$\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \end{bmatrix} = \begin{bmatrix} -1 & 2 & -2 \\ 0 & 1 & -2 \\ 2 & -1 & 0 \end{bmatrix}.$$

By elementary row operations on the augmented matrix, we have

$$\left[\begin{array}{cc|ccc} -1 & 1 & -1 & 2 & -2 \\ 0 & 1 & 0 & 1 & -2 \\ -3 & 1 & 2 & -1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|ccc} 1 & -1 & 1 & -2 & 2 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & -2 & 5 & -7 & 6 \end{array}\right] \Rightarrow \left[\begin{array}{cc|ccc} 1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 & -2 \\ 0 & 0 & 5 & -5 & 2 \end{array}\right],$$

which is inconsistent. Thus there is no such a matrix A and then no such a linear transformation.

Prob. 3:

"$\Rightarrow$" Since T is linear, for all scalars α and all vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, we have

$$T(\alpha\mathbf{x} - \mathbf{y}) = T(\alpha\mathbf{x} + (-1)\mathbf{y}) = \alpha T(\mathbf{x}) + (-1)T(\mathbf{y}) = \alpha T(\mathbf{x}) - T(\mathbf{y}).$$

"$\Leftarrow$" We first note that

$$T(\mathbf{0}) = T(1\mathbf{0} - \mathbf{0}) = 1T(\mathbf{0}) - T(\mathbf{0}) = T(\mathbf{0}) - T(\mathbf{0}) = \mathbf{0} \tag{1}$$

and

$$T(-\mathbf{y}) = T(0\mathbf{0} - \mathbf{y}) = 0T(\mathbf{0}) - T(\mathbf{y}) = -T(\mathbf{y}). \tag{2}$$

Now for all scalars α and all vectors $\mathbf{x}$ in $\mathbb{R}^n$, we have

$$T(\alpha\mathbf{x}) = T(\alpha\mathbf{x} - \mathbf{0}) = \alpha T(\mathbf{x}) - T(\mathbf{0}) = \alpha T(\mathbf{x}) - \mathbf{0} = \alpha T(\mathbf{x}), \tag{3}$$

where $T(\mathbf{0}) = \mathbf{0}$ by (1). And then for all scalars α, β and all vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$, we have

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = T(\alpha\mathbf{x} - (-(\beta\mathbf{y}))) = \alpha T(\mathbf{x}) - T(-(\beta\mathbf{y})) = \alpha T(\mathbf{x}) - (-T(\beta\mathbf{y})),$$

where $T(-(\beta\mathbf{y})) = -T(\beta\mathbf{y})$ by (2), and then

$$T(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha T(\mathbf{x}) + T(\beta\mathbf{y}) = \alpha T(\mathbf{x}) + \beta T(\mathbf{y}),$$

where $T(\beta\mathbf{y}) = \beta T(\mathbf{y})$ by (3).

Prob. 4:

By elementary row operations, we have

$$\left[\begin{array}{cc|c} a & 0 & 0 \\ c & d & 0 \\ -1 & f & 0 \end{array}\right] \Rightarrow \left[\begin{array}{cc|c} 1 & -f & 0 \\ 0 & af & 0 \\ 0 & d+cf & 0 \end{array}\right].$$

The linear transformation $T : \mathbf{x} \mapsto A\mathbf{x}$ is not one to one if and only if $\text{Ker}(A) \neq \{\mathbf{0}\}$ if and only if

$$af = 0 \text{ and } d + cf = 0$$

Thus the linear transformation $T : \mathbf{x} \mapsto A\mathbf{x}$ is one to one if and only if either $af \neq 0$ or $d + cf \neq 0$. For example, with $a = f = 1$ and c, d arbitrary, $T : \mathbf{x} \mapsto A\mathbf{x}$ is one to one.

Prob. 5:

With proof by contrapositive, we show that if $m > n$, then any linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ is not a surjective.

Consider a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ from $\mathbb{R}^n$ to $\mathbb{R}^m$ with

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}.$$

T is surjective if and only if $T(\mathbb{R}^n) = \mathbb{R}^m$, i.e., given any vector $\mathbf{y} = (y_1, y_2, \cdots, y_m)$ in $\mathbb{R}^m$, there is a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ in $\mathbb{R}^n$ such that $T(\mathbf{x}) = A\mathbf{x} = \mathbf{y}$. Since

$m > n$, there are at most n pivot positions in A. By elementary row operations on the augmented matrix, we must have

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & y_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & y_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & y_m \end{array}\right] \Rightarrow \left[\begin{array}{c|c} U & \begin{matrix} y'_1 \\ \vdots \\ y'_n \end{matrix} \\ \hline \mathbf{0} & \begin{matrix} y'_{n+1} \\ \vdots \\ y'_m \end{matrix} \end{array}\right]$$

where $\begin{bmatrix} U \\ \mathbf{0} \end{bmatrix}$ is in the reduced row echelon form of A with U an $n \times n$ matrix and $\mathbf{0}$ the $(m-n) \times n$ zero matrix and $y'_1, y'_2, \cdots, y'_m$ are linear combinations of $y_1, y_2, \cdots, y_m$. It is clear that the linear system $A\mathbf{x} = \mathbf{y}$ is not always consistent for any given $\mathbf{y}$ in $\mathbb{R}^m$. Thus we conclude that T is not surjective.

Prob. 6:

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a finite linearly dependent indexed subset of $\mathbb{R}^n$. Then there is a linear relation in S

$$\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = 0$$

with $\alpha_1, \alpha_2, \ldots, \alpha_k$ not all zeros. Now the image $T(S)$ of S under T is the indexed subset $\{T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_k)\}$ of $\mathbb{R}^m$. Since

$$\mathbf{0} = T(\mathbf{0}) = T(\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k) = \alpha_1 T(\mathbf{v}_1) + \alpha_2 T(\mathbf{v}_2) + \cdots + \alpha_k T(\mathbf{v}_k),$$

$T(S)$ is a linearly dependent indexed set.

Prob. 7:

A linear transformation may not map a finite linearly independent indexed set into a linearly independent indexed set. For example, consider the zero map 0 which maps all vectors $\mathbf{x}$ in $\mathbb{R}^n$ to the zero vector $\mathbf{0}$ in $\mathbb{R}^m$. Then for any finite linearly independent indexed subset $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ of $\mathbb{R}^n$, we have the image $0(S) = \{\mathbf{0}, \mathbf{0}, \ldots, \mathbf{0}\}$ which is clearly a linearly dependent indexed set.