EECS205003 Linear Algebra, Fall 2020

Quiz # 4, Solutions

Prob. 1:

Let $\mathbf{u} \in \text{Span}(U)$. Then there are $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ in U and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{R}$ for some $n \geq 1$ such that

$$\mathbf{u} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \cdots + \alpha_n \mathbf{u}_n.$$

Since $U \subseteq \text{Span}(V)$, each $\mathbf{u}_i$ is in $\text{Span}(V)$ so that there are $\mathbf{v}_{i1}, \mathbf{v}_{i2}, \ldots, \mathbf{v}_{im_i}$ in V and $\beta_{i1}, \beta_{i2}, \ldots, \beta_{im_i}$ in $\mathbb{R}$ for some $m_i \geq 1$ such that

$$\mathbf{u}_i = \beta_{i1}\mathbf{v}_{i1} + \beta_{i2}\mathbf{v}_{i2} + \cdots + \beta_{im_i}\mathbf{v}_{im_i}.$$

Now we have

$$\mathbf{u} = \sum_{i=1}^{n} \alpha_i \sum_{j=1}^{m_i} \beta_{ij}\mathbf{v}_{ij} = \sum_{i=1}^{n}\sum_{j=1}^{m_i} \alpha_i\beta_{ij}\mathbf{v}_{ij}$$

which shows that $\mathbf{u}$ is a linear combination of $\mathbf{v}_{ij}$'s and then $\mathbf{u} \in \text{Span}(V)$. We conclude that $\text{Span}(U) \subseteq \text{Span}(V)$.

Prob. 2:

Since $U \subseteq \text{Span}(U)$, we have $\text{Span}(U) \subseteq \text{Span}(\text{Span}(U))$. Also since $\text{Span}(U) \subseteq \text{Span}(U)$, we have $\text{Span}(\text{Span}(U)) \subseteq \text{Span}(U)$ by the result of Problem 1. We conclude that $\text{Span}(\text{Span}(U)) = \text{Span}(U)$.

Prob. 3:

1. Yes. The span of an empty set is also an empty set.
2. Yes. The span of $\{\mathbf{0}\}$ is $\{\mathbf{0}\}$ which contains one and only one vector.
3. No. Since $\text{Span}(\{ (1, 1, 0), (1, 0, 0) \}) \subseteq \text{Span}(\{ (0, 1, 0), (1, 0, 0) \})$, but $\{ (1, 1, 0), (1, 0, 0) \} \not\subseteq \{ (0, 1, 0), (1, 0, 0) \}$.
4. Yes. Let $\mathbf{s} \in \text{Span}(S)$. Then there are $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n$ in S and $\alpha_1, \alpha_2, \ldots, \alpha_n$ in $\mathbb{R}$ for some $n \geq 1$ such that $\mathbf{s} = \alpha_1\mathbf{s}_1 + \alpha_2\mathbf{s}_2 + \cdots + \alpha_n\mathbf{s}_n$. Since $S \subseteq T$, $\mathbf{s}_1, \mathbf{s}_2, \ldots, \mathbf{s}_n \in T$, so $s \in \text{Span}(T)$. We conclude that $\text{Span}(S) \subseteq \text{Span}(T)$.

Prob. 4:

Note that

$$\left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 3 & -2 & -1 & b \\ 1 & 2 & 5 & c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & 2 & -3a+b \\ 0 & 3 & 6 & -a+c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & -1 & a \\ 0 & 1 & 2 & -3a+b \\ 0 & 0 & 0 & 8a-3b+c \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & -2a+b \\ 0 & 1 & 2 & -3a+b \\ 0 & 0 & 0 & 8a-3b+c \end{array}\right].$$

Thus the vector (a,b,c) is in the span of the set $\{(1,3,1),(-1,-2,2),(-1,-1,5)\}$ of three vectors if and only if $8a-3b+c=0$. For any values of a,b,c not satisfying the equation $8a-3b+c=0$, the vector (a,b,c) is not in the span of $\{(1,3,1),(-1,-2,2),(-1,-1,5)\}$. For example, $(a,b,c)=(1,1,1)$ is not in the span.

Prob. 5:

A point $\mathbf{x}=(x_1,x_2,x_3,x_4)\in\mathbb{R}^4$ is in the plane if and only if there exist $s,t\in\mathbb{R}$ such that

$$\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ -2 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} x_1+1 \\ x_2-2 \\ x_3 \\ x_4+1 \end{bmatrix} \quad\Leftrightarrow\quad \left[\begin{array}{cc|c} 1 & 1 & x_1+1 \\ 1 & 0 & x_2-2 \\ -2 & -1 & x_3 \\ -1 & 0 & x_4+1 \end{array}\right] \text{ is consistent}$$

$$\Leftrightarrow\quad \left[\begin{array}{cc|c} 1 & 1 & x_1+1 \\ 0 & -1 & -x_1+x_2-3 \\ 0 & 1 & 2x_1+x_3+2 \\ 0 & 1 & x_1+x_4+2 \end{array}\right] \text{ is consistent}$$

$$\Leftrightarrow\quad \left[\begin{array}{cc|c} 1 & 0 & x_2-2 \\ 0 & -1 & -x_1+x_2-1 \\ 0 & 0 & x_1+x_2+x_3-1 \\ 0 & 0 & x_2+x_4-1 \end{array}\right] \text{ is consistent.}$$

Therefore a point $\mathbf{x}=(x_1,x_2,x_3,x_4)$ is in the plane if and only if $\begin{cases} x_1+x_2+x_3=1, \\ x_2+x_4=1. \end{cases}$

Prob. 6:

(a) Since

$$[A\,|b\,] = \left[\begin{array}{ccccc|c} -2 & 1 & 0 & 0 & -1 & -1 \\ 1 & -2 & -2 & -1 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 & -1 & -1 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccccc|c} 1 & -2 & -2 & -1 & 0 & 0 \\ 0 & -3 & -4 & -2 & -1 & -1 \end{array}\right] \Rightarrow \left[\begin{array}{ccccc|c} 1 & 0 & 2/3 & 1/3 & 2/3 & 2/3 \\ 0 & 1 & 4/3 & 2/3 & 1/3 & 1/3 \end{array}\right],$$

we have

$$\begin{bmatrix} 1 & 0 & 2/3 & 1/3 & 2/3 \\ 0 & 1 & 4/3 & 2/3 & 1/3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix}$$

and then

$$
\begin{aligned}
x_1 &= \frac{2}{3} - \frac{2}{3}x_3 - \frac{1}{3}x_4 - \frac{2}{3}x_5, \\
x_2 &= \frac{1}{3} - \frac{4}{3}x_3 - \frac{2}{3}x_4 - \frac{1}{3}x_5.
\end{aligned}
$$

Thus the solution set of the linear system $A\mathbf{x} = \mathbf{b}$ consists of vectors of the form

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2/3 \\ -4/3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1/3 \\ -2/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2/3 \\ -1/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}.
$$

Let $\mathbf{u} = (2/3, 1/3, 0, 0, 0)$, $\mathbf{v}_1 = (-2/3, -4/3, 1, 0, 0)$, $\mathbf{v}_2 = (-1/3, -2/3, 0, 1, 0)$ and $\mathbf{v}_3 = (-2/3, -1/3, 0, 0, 1)$. We have

$$
H = \{\mathbf{u} + t_1 \cdot \mathbf{v}_1 + t_2 \cdot \mathbf{v}_2 + t_3 \cdot \mathbf{v}_3 | t_1, t_2, t_3 \in \mathbb{R}\},
$$

which is a parametric representation of H.

(b) Since such a parametric representation of H has 3 free variables, the dimension of H is 3.

Prob. 7:

Let $H = \{\mathbf{u} + t_1 \cdot \mathbf{v}_1 + \cdots + t_k \cdot \mathbf{v}_k | t_1, \cdots, t_k \in \mathbb{R}\}$ be a k-dimensional affine space in $\mathbb{R}^n$, where $\mathbf{u}, \mathbf{v}_1, \cdots, \mathbf{v}_k \in \mathbb{R}^n$ and $\{\mathbf{v}_1 \cdots \mathbf{v}_k\}$ is a linearly independent set. A point $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is in H if and only if there exist $t_1, \ldots, t_k \in \mathbb{R}$ such that $\mathbf{x} = \mathbf{u} + t_1\mathbf{v}_1 + \cdots + t_k\mathbf{v}_k$, i.e., the linear system

$$
\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix} \begin{bmatrix} t_1 \\ t_2 \\ \vdots \\ t_k \end{bmatrix} = [\mathbf{x} - \mathbf{u}]
$$

is consistent. Let $A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k \end{bmatrix}$. Since the columns $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of A form a linearly independent set, there is a pivot position in each column of $\mathbf{A}$ by **Theorem 1.3.13**. Thus the reduced row echelon form of the augmented matrix $[\mathbf{A}|\mathbf{x} - \mathbf{u}]$ must be

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_k & \mathbf{x}-\mathbf{u} \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & \cdots & 0 & -b_{10}+b_{11}x_1+b_{12}x_2+\cdots+b_{1n}x_n \\ 0 & 1 & \cdots & 0 & -b_{20}+b_{21}x_1+b_{22}x_2+\cdots+b_{2n}x_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -b_{k0}+b_{k1}x_1+b_{k2}x_2+\cdots+b_{kn}x_n \\ \hline 0 & 0 & \cdots & 0 & -b_{k+10}+b_{k+11}x_1+b_{k+12}x_2+\cdots+b_{k+1n}x_n \\ 0 & 0 & \cdots & 0 & -b_{k+20}+b_{k+21}x_1+b_{k+22}x_2+\cdots+b_{k+2n}x_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & -b_{n0}+b_{n1}x_1+b_{n2}x_2+\cdots+b_{nn}x_n \end{array}\right]$$

by elementary row operations, where $b_{ij}, 1 \le i \le n, 0 \le j \le n$, are scalars. Now the linear system $A\mathbf{t} = \mathbf{x} - \mathbf{u}$ is consistent if and only if the last $n-k$ entries of the last column in the reduced row echelon form of the augmented matrix $[A|\mathbf{x}-\mathbf{u}]$ are all zeros, i.e.,

$$\begin{aligned} b_{k+11}x_1 + b_{k+12}x_2 + \cdots + b_{k+1n}x_n &= b_{k+10}, \\ b_{k+21}x_1 + b_{k+22}x_2 + \cdots + b_{k+2n}x_n &= b_{k+20}, \\ &\vdots \\ b_{n1}x_1 + b_{n2}x_2 + \cdots + b_{nn}x_n &= b_{n0}. \end{aligned}$$

We conclude that a point $\mathbf{x} = (x_1, \cdots, x_n) \in \mathbb{R}^n$ is in H if and only if it is in the solution set of the above system of $(n-k)$ linear equations with n unknowns.