EECS205003 Linear Algebra, Fall 2020

Quiz # 3, Solutions

Prob. 1: By elementary row operations, we have

$$\begin{bmatrix} -1 & 2 & 0 & 1 \\ 2 & -1 & 3 & -8 \\ 1 & -1 & 1 & -3 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 3 & 3 & -6 \\ 0 & 1 & 1 & -2 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & -2 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then, the rank of this matrix is 2.
By the definition of kernel, $\mathrm{Ker}(A) = \{\mathbf{x} \in \mathbb{R}^4 : A\mathbf{x} = \mathbf{0}\}$. Now

$$\begin{bmatrix} 1 & 0 & 2 & -5 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

Thus $\mathrm{Ker}(A) = \mathrm{Span}\{(-2,-1,1,0),(5,2,0,1)\}$.

Prob. 2:

The answer is yes.
To find a possible inverse of A, we solve the matrix equation $AX = I_{3\times 3}$ for an unknown 3×3 matrix X as follows:

$$\left[\begin{array}{ccc|ccc} -1 & 2 & -1 & 1 & 0 & 0 \\ 2 & 0 & 3 & 0 & 1 & 0 \\ 1 & -1 & 2 & 0 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} -1 & 2 & -1 & 1 & 0 & 0 \\ 0 & 4 & 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & -2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 4 & 1 & 2 & 1 & 0 \end{array}\right] \Rightarrow$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & -3 & -2 & 1 & -4 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 2 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \end{array}\right] \Rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -2 \\ 0 & 1 & 0 & \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ 0 & 0 & 1 & \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \end{array}\right].$$

Thus we have $X = \begin{bmatrix} -1 & 1 & -2 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \end{bmatrix}$ with $AX = I$. To verify that X is the inverse of A, we should check whether $XA = I$ as follows:

$$XA = \begin{bmatrix} -1 & 1 & -2 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ 2 & 0 & 3 \\ 1 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

We conclude that A is invertible and $X = \begin{bmatrix} -1 & 1 & -2 \\ \frac{1}{3} & \frac{1}{3} & -\frac{1}{3} \\ \frac{2}{3} & -\frac{1}{3} & \frac{4}{3} \end{bmatrix}$ is the inverse of A.

Prob. 3:

The answer is yes. Consider the set of vectors

$$\{(1,0,-1,2),(3,0,1,4),(0,0,-2,1)\} = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}.$$

This set is linearly dependent if and only if there is a nontrivial linear relation $a\mathbf{v}_1 + b\mathbf{v}_2 + c\mathbf{v}_3 = \mathbf{0}$ with a, b, c not all zeros. The linear relation can be rewritten as a linear system,

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 0 \\ -1 & 1 & -2 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Now by elementary row operations, we have

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 1 & -2 & 0 \\ 2 & 4 & 1 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 4 & -2 & 0 \\ 0 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{3}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus the original linear system is row equivalent to the following linear system,

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} \\ 0 & 1 & -\frac{1}{2} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

which shows that c is a free variable. Thus, the set of vectors is linearly dependent, and a nontrivial linear relation for this set of vectors is $-3(1,0,-1,2) + (3,0,1,4) + 2(0,0,-2,1) = (0,0,0,0)$

Prob. 4:

Consider an arbitrary linear relation of vectors in the set $\{\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3, -2\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_3\}$,

$$\alpha(\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3) + \beta(-2\mathbf{v}_2) + \gamma(\mathbf{v}_1 - \mathbf{v}_3) = \mathbf{0}.$$

Then we have

$$(\alpha + \gamma)\mathbf{v}_1 + (-\alpha - 2\beta)\mathbf{v}_2 + (\alpha - \gamma)\mathbf{v}_3 = \mathbf{0}.$$

Since the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent, we must have

$$\alpha + \gamma = -\alpha - 2\beta = \alpha - \gamma = 0$$

and then $\alpha = \beta = \gamma = 0$. We conclude that the set $\{\mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3, -2\mathbf{v}_2, \mathbf{v}_1 - \mathbf{v}_3\}$ of vectors is linearly independent.

Prob. 5:

Let $\mathbf{r}_i$ be the i-th row of A_n. We will show that for all $i > 2$, $\mathbf{r}_i$ is a linear combination of $\mathbf{r}_1$ and $\mathbf{r}_2$. Note that $\mathbf{r}_1 = [1, 2, \cdots, n]$ and $\mathbf{r}_2 = [n+n, n+n-1, \cdots, n+1]$. In general, there are two cases for $\mathbf{r}_i$. Let $a > 1$ be a positive integer. In the first case, $\mathbf{r}_i = [an+1, an+2, \cdots, an+n]$, then $\mathbf{r}_i = an \cdot \dfrac{\mathbf{r}_1 + \mathbf{r}_2}{2n+1} + \mathbf{r}_1$. In the second case, $\mathbf{r}_i = [an+n, an+n-1, \cdots, an+1]$, then $\mathbf{r}_i = (a-1)n \cdot \dfrac{\mathbf{r}_1 + \mathbf{r}_2}{2n+1} + \mathbf{r}_2$. Since $\mathbf{r}_i$ is a linear combination of $\mathbf{r}_1$ and $\mathbf{r}_2$ for all $i > 2$, we have

$$A_n \sim \begin{bmatrix} 1 & 2 & \cdots & n \\ 2n & 2n-1 & \cdots & n+1 \\ 0 & 0 & 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & \cdots & n \\ 0 & -2n-1 & \cdots & n+1-2n^2 \\ 0 & 0 & 0 & 0 \\ \vdots & & & \vdots \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Therefore $\text{Rank}(A_n) = \begin{cases} 1, & n = 1, \\ 2, & n > 1. \end{cases}$

Prob. 6:

According to Corollary 1 in Section 1.3 of the textbook on page 65, the rank of a matrix is the number of pivots in its reduced row echelon form, which is the same as the number of pivot positions in the matrix.

(1) The answer to the first question is yes. If two matrices are row equivalent to each other, then they have the same reduced row echelon form so that they have the same number of pivots. Thus, they have the same rank.

(2) The answer to the second question is no. For example, consider the following two matrices in reduced row echelon form:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is clear that both of them have the same rank 2. But they are not row equivalent to each other.

Prob. 7:

We will provide two solutions.

Solution I:

Let $\mathbf{A}$ be an $m \times n$ matrix and $\mathbf{U}$ in the reduced echelon form of $\mathbf{A}$. By **Lemma 1.3.A** in the class lecture, the set of column vectors of $\mathbf{A}$ is linearly independent if and only if $\text{Ker}(\mathbf{A}) = \{\mathbf{0}\}$. Since $\mathbf{A}$ and $\mathbf{U}$ are row equivalent, $\text{Ker}(\mathbf{A}) = \text{Ker}(\mathbf{U})$

by **Theorem 1.3.5** of the textbook. Thus the set of column vectors of $\mathbf{A}$ is linearly independent if and only if $\text{Ker}(\mathbf{U}) = \{\mathbf{0}\}$ if and only if the homogeneous linear system $\mathbf{Ux} = \mathbf{0}$ only has the trivial solution if and only if all variables $x_1, x_2, \dots, x_n$ are dependent if and only if each column of $\mathbf{U}$ has a pivot if and only if each column in $\mathbf{A}$ has a pivot position.

Solution II:

From **Theorem 1.3.8** of the textbook, the following two properties of an $m \times n$ matrix $\mathbf{A}$ are equivalent to each other:

(i) At least one column of $\mathbf{A}$ has no pivot position.
(ii) The system $\mathbf{Ax} = \mathbf{0}$ has some nontrivial solutions.

Equivalently, each column in $\mathbf{A}$ has a pivot position $\Leftrightarrow$ the linear system $\mathbf{Ax} = \mathbf{0}$ only has the trivial solution. But the column vectors of a matrix form a linearly independent set $\Leftrightarrow$ the linear system $\mathbf{Ax} = \mathbf{0}$ only has the trivial solution. We conclude that the column vectors of a matrix form a linearly independent set $\Leftrightarrow$ each column in $\mathbf{A}$ has a pivot position.