EECS205003 Linear Algebra, Fall 2020

Quiz # 1, Solutions

Prob. 1: We have

$$\begin{bmatrix} -2 & 0 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 4 & 3 & 2 & 4 \\ 2 & 3 & 1 & 3 \\ 3 & 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 3 & 0 & 8 \\ 0 & 3 & 0 & 5 \\ 0 & 0 & -\frac{1}{2} & 2 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & 4 \end{bmatrix} \to$$

$$\begin{bmatrix} -2 & 0 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 10 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} -2 & 0 & -1 & 2 \\ 0 & 3 & -1 & 2 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & -6 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

where the last two matrices are in row echelon form.

Prob. 2: We have

$$\begin{bmatrix} 0 & 2 & 2 & 0 & 0 \\ -3 & -1 & -2 & -1 & 1 \\ 3 & 1 & 0 & -1 & 0 \\ -3 & -1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ -3 & -1 & -2 & -1 & 1 \\ -3 & -1 & 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 1 & 0 & -1 & 0 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & -2 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to$$

$$\begin{bmatrix} 3 & 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & -1 & -1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 & 0 & \frac{-1}{2} \\ 0 & 1 & 0 & -1 & \frac{1}{2} \\ 0 & 0 & 1 & 1 & \frac{-1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Prob. 3:

1. Consider a matrix $A = [a_{ij}]$ with integer entries only. In the process of Gaussian elimination, a swap row operation is used to establish a pivot. A matrix with integer entries only is also a matrix with integer entries only after a swap operation. And a replacement row operation is used to eliminate a nonzero entry a_{jk} below a pivot a_{ik}, which may result in non-integer entries. Instead, a scaling row operation $S_j(a_{ik})$ is used to multiply the jth row by a_{ik}, which results only integer entries, and then a replacement row operation $E_{ji}(-a_{jk})$ is used to eliminate the nonzero entry $a_{ik}a_{jk}$ below the pivot a_{ik}, which results only integer entries. For example, we have

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1} & a_{j2} & a_{j3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \to \begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{j1}a_{i1} & a_{j2}a_{i1} & a_{j3}a_{i1} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix} \to$$

$$\begin{bmatrix} \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{i1} & a_{i2} & a_{i3} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & a_{j2} \times a_{i1} - a_{i2} \times a_{j1} & a_{j3} \times a_{i1} - a_{i3} \times a_{j1} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}.$$

In this version of Gaussian elimination, a matrix with integer entries only will be row equivalent to a matrix in row echelon form having only integer entries.

2. No. A counterexample is

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{1}{2} \end{bmatrix}.$$

Prob. 4:

1. We claim that $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$. Here is a proof. If $f(\bigcap_i A_i) = \varnothing$, then of course we have $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$. Assume $f(\bigcap_i A_i) \neq \varnothing$. Let $y \in f(\bigcap_i A_i)$. Then there is an $x \in \bigcap_i A_i$ such that $f(x) = y$. Since $x \in \bigcap_i A_i$, $x \in A_i$ for all i. This implies that $y \in f(A_i)$ for all i and then $y \in \bigcap_i f(A_i)$. We conclude that $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$.

2. In general, we cannot have $f(\bigcap_i A_i) = \bigcap_i f(A_i)$. For example, let $A = \{1, 2, 3\}$, $B = \{a, b, c\}$, and f be a function from A to B with $f(1) = a$, $f(2) = b$, $f(3) = a$. Let $A_1 = \{1, 2\}$ and $A_2 = \{3\}$. Then $A_1 \cap A_2 = \emptyset$ so that $f(A_1 \cap A_2) = f(\emptyset) = \emptyset$. But we have $f(A_1) \cap f(A_2) = \{a, b\} \cap \{a\} = \{a\} \neq \emptyset = f(A_1 \cap A_2)$.

Thus the relationship between $f(\bigcap_i A_i)$ and $\bigcap_i f(A_i)$ is $f(\bigcap_i A_i) \subseteq \bigcap_i f(A_i)$.

Prob. 5:

No.

P	Q	$P \to Q$	$\neg P$	$\neg Q$	$\neg P \to \neg Q$
F	F	T	T	T	T
F	T	T	T	F	F
T	F	F	F	T	T
T	T	T	F	F	T

Prob. 6:

When $n = 1$, $1 + 1 + 1 = 3$ is an odd number. Assume that $n = k$ is true, i.e., $k^2 + k + 1 = 2a + 1$ for some $a \in \mathbb{Z}$. Consider $n = k + 1$. We have

$$(k + 1)^2 + (k + 1) + 1 = (2a + 1) + 2k + 2,$$

which is also an odd number. Therefore by induction, we conclude that $n^2 + n + 1$ is an odd number $\forall n \in \mathbb{N}$.

Prob. 7:

Let $A = \frac{a}{b}, B = \frac{c}{d}, a, b, c, d \in \mathbb{Z}$ with $b, d \neq 0$ be two arbitrary rational numbers. Also let C be an arbitrary irrational number. It is clear that $C \neq 0$.

1. Correct. This is because that $A + B = \frac{ad+bc}{bd}$ with $ad + bc, bd \in \mathbb{Z}$ and $bd \neq 0$.
2. Correct. Suppose that $D = A + C$ is rational. Then we have $C = D + (-A)$ to be rational by 1 in above, a contradiction. Thus D must be irrational.