EE205003 Linear Algebra, 2020 Fall Semester

Final Examination

DATE: Jan. 13th, 2021

1. Please solve the following linear systems if consistent or find a least squares solution if inconsistent.

   (a) (5%) $\begin{bmatrix} 2 & -1 \\ 0 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 1 \\ 1 \end{bmatrix}$.

   (b) (5%) $\begin{bmatrix} 1 & -1 & 0 \\ 0 & -2 & 1 \\ -2 & 2 & 3 \\ 4 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 4 \\ 4 \end{bmatrix}$.

2. The trace $\text{tr}(A)$ of an $n \times n$ (real or complex) matrix A is defined as the sum of its diagonal entries, i.e.,

$$\text{tr}(A) = a_{11} + a_{22} + \cdots + a_{nn}.$$

   (a) (5%) Please show that if $p(\lambda) = (-1)^n \lambda^n + p_{n-1}\lambda^{n-1} + \cdots + p_1\lambda + p_0$ is the characteristic polynomial of A, then

$$\text{tr}(A) = (-1)^{n-1} p_{n-1} \text{ and } \text{Det}(A) = p_0.$$

   (b) (5%) Please show that if A and B are similar matrices, then $\text{tr}(A) = \text{tr}(B)$ and $\text{Det}(A) = \text{Det}(B)$.

3. (10%) Let A be an $m \times n$ complex matrix. Please show that AA^H is invertible if and only if the rank of A is m.

4. (10%) Let $S = \{(1, 0, -2, 1, 0), (0, 1, 0, 1, 0), (1, 0, 3, -1, 1)\}$ be a subset of $\mathbb{R}^5$. Please find the orthogonal complement $S^\perp$ of S in $\mathbb{R}^5$.

5. (10%) Find an orthogonal basis for $\text{Span}(S)$ where $S = \{(1 - i, 3, i, 0), (2, 2i, 0, 1 + i)\}$ is a subset of $\mathbb{C}^4$.

6. (10%) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in an inner product space V over F, $F = \mathbb{R}$ or $\mathbb{C}$. Please show that if $\langle \mathbf{u}, \mathbf{x} \rangle = \langle \mathbf{v}, \mathbf{x} \rangle$ for all $\mathbf{x} \in V$, then $\mathbf{u} = \mathbf{v}$.

7. (10%) Please show that any 2×2 real orthogonal matrix is either $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ or $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ for some real θ.

8. (10%) Define an inner product in $\mathbb{C}^n$ by

$$\langle \mathbf{x}, \mathbf{y} \rangle_o \triangleq q_1 x_1 \bar{y}_1 + q_2 x_2 \bar{y}_2 + \cdots + q_n x_n \bar{y}_n$$

   for all $\mathbf{x}, \mathbf{y} \in \mathbb{C}^n$, where $q_1, q_2, \ldots, q_n$ are positive scalars. Define a linear operator T on $\mathbb{C}^n$ by $T(\mathbf{x}) = A\mathbf{x}$ with A a complex $n \times n$ matrix. Please find the adjoint of T. (Hint: Express $\langle \mathbf{x}, \mathbf{y} \rangle_o$ as $\mathbf{y}^H \Lambda \mathbf{x}$, where Λ is the diagonal matrix with diagonal entries $q_1, q_2, \ldots, q_n$.)

9. (10%) Let L be a linear operator on a finite-dimensional inner product space V over $\mathbb{C}$. Let A be the matrix representation of T with respective to an orthonormal basis B for V, i.e., $[T]_B = A$. Please show that T is self-adjoint if and only if A is Hermitian, i.e., $A^H = A$.

10. Please justify whether each of the following complex matrices A is unitarily diagonalizable, i.e., $D = U^H AU$, where U is unitary and D is diagonal. If yes, please find a unitary matrix U to diagonalize the matrix A. If no, please give a reason, otherwise no credits.

(a) (5%) $A = \begin{bmatrix} 3 & 2i & 1-i \\ 1-i & 0 & 0 \\ -2i & 1+i & 3 \end{bmatrix}$.

(b) (5%) $A = \begin{bmatrix} 1-i & 2i & -2 \\ 2i & i & 4 \\ -6i & -4 & 10-2i \end{bmatrix}$.