EECS205003 Linear Algebra, Fall 2020

Final Exam Solutions

Prob. 1:

(a)

$$\left[\begin{array}{cc|c} 2 & -1 & 4 \\ 0 & 2 & 4 \\ 1 & 0 & 1 \\ -2 & 1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 2 & 4 \\ 0 & 1 & 3 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 8 \\ 0 & 0 & 5 \end{array}\right]$$

This system is inconsistent. For a least-squares solution, consider the system $A^T Ax = A^T b$:

$$\begin{bmatrix} 2 & 0 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 0 & 2 \\ 1 & 0 \\ -2 & 1 \end{bmatrix} x = \begin{bmatrix} 2 & 0 & 1 & -2 \\ -1 & 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 4 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 9 & -4 \\ -4 & 6 \end{bmatrix} x = \begin{bmatrix} 7 \\ 5 \end{bmatrix}$$

$$\Rightarrow \left[\begin{array}{cc|c} 9 & -4 & 7 \\ -4 & 6 & 5 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & -\frac{4}{9} & \frac{7}{9} \\ 0 & \frac{38}{9} & \frac{73}{9} \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & \frac{31}{19} \\ 0 & 1 & \frac{73}{38} \end{array}\right]$$

A least-squares solution is $\begin{bmatrix} \frac{31}{19} \\ \frac{73}{38} \end{bmatrix}$.

(b)

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & -2 & 1 & 1 \\ -2 & 2 & 3 & 4 \\ 4 & 0 & -1 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 0 & 1 \\ 0 & 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & 3 & 6 \\ 0 & 4 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{3}{2} \\ 0 & 1 & 0 & \frac{1}{2} \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This system is consistent and the solution is $\begin{bmatrix} \frac{3}{2} \\ \frac{1}{2} \\ 2 \end{bmatrix}$.

Prob. 2:

(a) Since $p(\lambda) = \text{Det}(A - \lambda I) = \text{Det} \begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix}$, the coefficients p_{n-1} of the term of degree $n - 1$ comes from the product

$$(a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda) = \cdots + (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} + (-1)^n \lambda^n$$

and then

$$p_{n-1} = (-1)^{n-1}(a_{11} + a_{22} + \cdots + a_{nn})$$

so that

$$tr(A) = a_{11} + a_{22} + \cdots + a_{nn} = (-1)^{n-1}p_{n-1}.$$

Also $p_0 = p(0) = \text{Det}(A - 0I) = \text{Det}(A)$.

(b) Since similar matrices have the same characteristic polynomial and then have the same p_{n-1} and p_0, we have

$$tr(A) = (-1)^{n-1}p_{n-1} = tr(B) \text{ and } \text{Det}(A) = p_0 = \text{Det}(B).$$

Prob. 3:

"$\Rightarrow$" Note that since AA^H is invertible, $\text{Ker}(AA^H) = \{\mathbf{0}\}$. Suppose $\mathrm{x} \in \text{Ker}(A^H)$, then $AA^H\mathbf{x} = A\mathbf{0} = \mathbf{0}$. We have $\mathbf{x} \in \text{Ker}(AA^H) = \{\mathbf{0}\}$. Therefore, $\text{Ker}(A^H) = \{\mathbf{0}\}$. Since $\text{Ker}(A^H) = \{\mathbf{0}\}$ and $\text{rank}(A^H) = \text{rank}(A)$, the rank of A is m.

"$\Leftarrow$" Note that since A^H has pivots in all of its columns, $\text{Ker}(A^H) = \{\mathbf{0}\}$. Suppose $\mathrm{x} \in \text{Ker}(AA^H)$, then $\mathbf{x}^H AA^H\mathbf{x} = (A^H\mathbf{x})^H A^H\mathbf{x} = \mathbf{0}$. We have $A^H\mathbf{x} = \mathbf{0}$ and $\mathbf{x} \in \text{Ker}(A^H) = \{\mathbf{0}\}$. Therefore, $\text{Ker}(AA^H) = \{\mathbf{0}\}$. Since $\text{Ker}(AA^H) = \{\mathbf{0}\}$ and AA^H is square, AA^H is invertible.

Prob. 4:

By definition in the textbook, the orthogonal complement set $S^\perp$ of S in $\mathbb{C}^3$ is the set

$$S^\perp = \{\mathbf{x} \mid \mathbf{x} \perp \mathrm{s} \text{ for all } \mathrm{s} \in S\}.$$

Since $S = \{(1,0,-2,1,0), (0,1,0,1,0), (1,0,3,-1,1)\}$, we have the equation

$$\begin{bmatrix} 1 & 0 & -2 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

where $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ in $\mathbb{R}^5$.

$$\left[\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 3 & -1 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & -2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 5 & -2 & 1 & 0 \end{array}\right] \to \left[\begin{array}{ccccc|c} 1 & 0 & 0 & \frac{1}{5} & \frac{2}{5} & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -\frac{2}{5} & \frac{1}{5} & 0 \end{array}\right]$$

Thus the orthogonal complement set $S^\perp$ of S is

$$\left\{ \mathbf{x} \;\middle|\; \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{1}{5} \\ -1 \\ \frac{2}{5} \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -\frac{2}{5} \\ 0 \\ -\frac{1}{5} \\ 0 \\ 1 \end{bmatrix} t,\ s, t \in \mathbb{R} \right\}.$$

Prob. 5:

Use the modified Gram-Schmidt algorithm. Let $\mathbf{u}_1 = (1-i,\ 3,\ i,\ 0), \mathbf{u}_2 = (2,\ 2i,\ 0,\ 1+i)$. An orthogonal basis for Span(S) consists of the following two vectors:

$$\begin{aligned}\mathbf{z}_1 &= \mathbf{u}_1 = (1-i,\ 3,\ i,\ 0),\\ \mathbf{z}_2 &= \mathbf{u}_2 - \text{proj}_{\mathbf{z}_1}\mathbf{u}_2 = (2,\ 2i,\ 0,\ 1+i) - (\frac{2+8i}{12})(1-i,\ 3,\ i,\ 0)\\ &= (\frac{7-3i}{6}, -\frac{1}{2}, \frac{4-i}{6}, 1+i).\end{aligned}$$

Prob. 6:

Suppose that $\mathbf{u} \neq \mathbf{v}$. Then $\mathbf{v} = \mathbf{u} + \mathbf{y}$ for some nonzero vector $\mathbf{y}$. Let $\mathbf{x} = \mathbf{y}$, then

$$\langle \mathbf{v}, \mathbf{y}\rangle = \langle \mathbf{u}, \mathbf{y}\rangle + \underbrace{\langle \mathbf{y}, \mathbf{y}\rangle}_{\neq 0} \neq \langle \mathbf{u}, \mathbf{y}\rangle,$$

a contradiction. We conclude that if $\langle \mathbf{u}, \mathbf{x}\rangle = \langle \mathbf{v}, \mathbf{x}\rangle$ for all $\mathbf{x} \in V$, then $\mathbf{u} = \mathbf{v}$.

Prob. 7:

For a unit vector $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$, since $x_1^2 + x_2^2 = 1$, we can express $\mathbf{x}$ as $(\cos\theta, \sin\theta)$ for some θ. Since all the columns of an orthogonal matrix are unit vectors, any 2×2 real orthogonal matrix A can be expressed as $A = \begin{bmatrix}\cos\theta & \cos\phi\\ \sin\theta & \sin\phi\end{bmatrix}$ for some real θ and ϕ. Since the columns of A are orthogonal to each other, we have

$$\cos\theta\cos\phi + \sin\theta\sin\phi = \cos(\theta-\phi) = \cos(\phi-\theta) = 0.$$

Therefore,

$$\phi = \left(\frac{1}{2} + n\right)\pi + \theta,\ n \in \mathbb{Z}.$$

If n is even, then $(\cos\phi, \sin\phi) = (-\sin\theta, \cos\theta)$. Otherwise, if n is odd, then $(\cos\phi, \sin\phi) = (\sin\theta, -\cos\theta)$. Therefore, any 2×2 real orthogonal matrix is either $\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ or $\begin{bmatrix}\cos\theta & \sin\theta\\ \sin\theta & -\cos\theta\end{bmatrix}$ for some real θ.

Prob. 8:

Let Λ be a diagonal matrix with diagonal entries $q_1, q_2, \cdots, q_n$, then $\langle \mathbf{x}, \mathbf{y}\rangle_o = \mathbf{y}^H\Lambda\mathbf{x}$.

$$\begin{aligned}\langle T(\mathbf{x}), \mathbf{y}\rangle_o =& \langle A\mathbf{x}, \mathbf{y}\rangle_o = \mathbf{y}^H\Lambda A\mathbf{x} = \mathbf{y}^H\Lambda A\Lambda^{-1}\Lambda\mathbf{x}\\ =&((\Lambda A\Lambda^{-1})^H\mathbf{y})^H\Lambda\mathbf{x} = \langle \mathbf{x}, (\Lambda A\Lambda^{-1})^H\mathbf{y}\rangle_o.\end{aligned}$$

Therefore the adjoint of T is $T^{\dagger}(\mathbf{x}) = (\Lambda A \Lambda^{-1})^H \mathbf{x} = \Lambda^{-1} A^H \Lambda \mathbf{x}$.

Prob. 9:

Let $B = \{\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_n\}$ be an orthonormal basis for V. The B-coordinate vectors of $\mathbf{x}, \mathbf{y} \in V$ will be denoted as

$$[\mathbf{x}]_B = (a_1, a_2, \dots, a_n) \text{ and } [\mathbf{y}]_B = (b_1, b_2, \dots, b_n).$$

Thus we have

$$\mathbf{x} = \sum_{i=1}^{n} a_i \mathbf{u}_i \text{ and } \mathbf{y} = \sum_{j=1}^{n} b_j \mathbf{u}_j.$$

The inner product of $\mathbf{x}$ and $\mathbf{y}$ in V is

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \sum_{i=1}^{n} a_i \mathbf{u}_i, \sum_{j=1}^{n} b_j \mathbf{u}_j \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} a_i \overline{b_j} \langle \mathbf{u}_i, \mathbf{u}_j \rangle$$
$$= \sum_{i=1}^{n} a_i \overline{b_i} = [\mathbf{y}]_B^H [\mathbf{x}]_B = \langle [\mathbf{x}]_B, [\mathbf{y}]_B \rangle,$$

where the last inner product is the standard inner product in $\mathbb{C}^n$. Note that $[T(\mathbf{x})]_B = [T]_B [\mathbf{x}]_B = A[\mathbf{x}]_B$ for all $\mathbf{x} \in V$.

Now T is self-adjoint if and only if $\langle T(\mathbf{x}), \mathbf{y} \rangle = \langle \mathbf{x}, T(\mathbf{y}) \rangle$ for all $\mathbf{x}$ and $\mathbf{y}$ in V if and only if $\langle [T(\mathbf{x})]_B, [\mathbf{y}]_B >= \langle [\mathbf{x}]_B, [T(\mathbf{y})]_B \rangle$ for all $\mathbf{x}$ and $\mathbf{y}$ in V if and only if $\langle A[\mathbf{x}]_B, [\mathbf{y}]_B \rangle = \langle [\mathbf{x}]_B, A[\mathbf{y}]_B \rangle$ for all $\mathbf{x}$ and $\mathbf{y}$ in V if and only if $[\mathbf{y}]_B^H A [\mathbf{x}]_B = [\mathbf{y}]_B^H A^H [\mathbf{x}]_B$ for all $\mathbf{x}$ and $\mathbf{y}$ in V if and only if $\mathbf{b}^H A \mathbf{a} = \mathbf{b}^H A^H \mathbf{a}$ for all $\mathbf{a}$ and $\mathbf{b}$ in $\mathbb{C}^n$ if and only if $A = A^H$, i.e., A is a Hermitian complex matrix.

Prob. 10:

(a) Note that

$$A^H = \begin{bmatrix} 3 & 1+i & 2i \\ -2i & 0 & 1-i \\ 1+i & 0 & 3 \end{bmatrix}.$$

Then we have

$$AA^H = \begin{bmatrix} 15 & 3+3i & 5+5i \\ 3-3i & 2 & 2+2i \\ 5-5i & 2-2i & 15 \end{bmatrix} \neq \begin{bmatrix} 15 & -2+8i & 3+3i \\ 2-8i & 6 & 1-5i \\ 3-3i & 1+5i & 11 \end{bmatrix} = A^H A$$

which shows that A is not a normal matrix. Thus A is not unitarily diagonalizable.

(b) Note that

$$A^H = \begin{bmatrix} 1+i & -2i & 6i \\ -2i & -i & -4 \\ -2 & 4 & 10+2i \end{bmatrix}.$$

Then we have

$$AA^H = \begin{bmatrix} 10 & -8-2i & -14-6i \\ -8+2i & 21 & 28+4i \\ -14+6i & 28-4i & 156 \end{bmatrix} \neq \begin{bmatrix} 42 & -22i & 10+50i \\ 22i & 21 & -40+8i \\ 10-50i & -40-8i & 124 \end{bmatrix} = A^H A$$

which shows that A is not a normal matrix. Thus A is not unitarily diagonalizable.