EE205003 Linear Algebra, 2020 Fall Semester

Midterm II

DATE: Dec. 16, 2020

1. (10%) Please calculate the determinant of the matrix

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 10 & -1 & 0 & 0 & 0 & 0 \\ -9 & 0 & 5 & 0 & 0 & 0 \\ 7 & -3 & 2 & -2 & 0 & 0 \\ -13 & 0 & -25 & 2 & 3 & -5 \\ 7 & 4 & -10 & 11 & 4 & -2 \end{bmatrix}.$$

2. (10%) Let A be an $n \times n$ invertible matrix. Please explain why in each column there is an entry that can be changed to make the matrix noninvertible.

3. (10%) Please show that an $n \times n$ matrix A is invertible if and only if $\mathrm{Det}(A) \neq 0$.

4. (5%) Consider

$$S = \{(1,-1,4,-2,3),(0,1,4,-1,2),(1,-2,0,3,1),(3,-8,-8,11,-1)\}$$

in $\mathbb{R}^5$. Please find a basis for Span(S).

5. Let $C = \{1, t, t^2, t^3\}$ be the ordered standard basis and $B = \{p_1(t), p_2(t), p_3(t), p_4(t)\}$ with $p_1(t) = t^3 - 2t^2 + 1, p_2(t) = -t^3 + 1, p_3(t) = t - 2, p_4(t) = 2t^2 - t + 1$ another ordered basis for $\mathbb{P}_3$.

   (a) (5%) Please find the transition matrix from the B-coordinates to the C-coordinate.

   (b) (5%) Please find the B-coordinate vector of $p(t) = t^3 + t^2 - t - 2$.

6. (10%) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ with $T(\mathbf{x}) = A\mathbf{x}$, where $A = \begin{bmatrix} -2 & 1 & -1 \\ -1 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Please find the matrix representation of T relative to the basis $B = \{(1,0,1),(2,-1,0),(-2,1,-1)\}$ for $\mathbb{R}^3$.

7. (10%) Let $B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $C = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Let V be the set of all 2×2 matrices A such that $CAB = O_{2\times 2}$. Please show that V is a subspace of $\mathcal{M}_{n\times n}$ and find $\mathrm{Dim}(V)$.

8. True or false. Please give reason(s), otherwise no credits.

   (a) (5%) An $n \times n$ matrix A and its transpose A^T have the same eigenvalues.

   (b) (5%) An invertible $n \times n$ matrix A may have an eigenvalue 0.

9. Consider a linear operator T on $\mathbb{R}^3$ defined by the matrix $A = \begin{bmatrix} -1 & 0 & 4 \\ 2 & -2 & -10 \\ -1 & 0 & 3 \end{bmatrix}$, i.e., $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

   (a) (10%) Find the eigenvalues of T and the corresponding eigenspaces.

   (b) (2%) What are the algebraic multiplicity and the geometric multiplicity of each eigenvalue of T.

   (c) (3%) Is it possible to find a basis B for $\mathbb{R}^3$ so that the matrix representation $[T]_B$ of T relative to B is diagonal? Explain your answer, otherwise no credits.

10. (10%) Let A be a diagonalizable $n \times n$ matrix, i.e., A is similar to a diagonal matrix D. If $A^2 + 2A - 15I_{n\times n} = 0_{n\times n}$, please show that the eigenvalues of A can only be 3 or -5.