EECS205003 Linear Algebra, Fall 2020

Midterm # 2, Solutions

Prob. 1:

$$1 \times (-1) \times 5 \times (-2) \times [3 \times (-2) - 4 \times (-5)] = 140.$$

Prob. 2:

For a matrix $A = [a_{ij}]$, we have

$$\mathrm{Det}(A) = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \mathrm{Det}(M^{ij})$$

for each column index j by Theorem 1 in Section 4.2 of the textbook. Fix a column j. Since A is invertible, we have $\mathrm{Det}(A) \neq 0$. There is a row i' such that $\mathrm{Det}(M^{i'j}) \neq 0$, otherwise $\mathrm{Det}(A) = 0$, a contradiction. Also since $\mathrm{Det}(A) \neq 0$, we have

$$(-1)^{i'+j} a_{i'j} \mathrm{Det}(M^{i'j}) \neq - \sum_{i=1, i \neq i'}^{n} (-1)^{i+j} a_{ij} \mathrm{Det}(M^{ij}).$$

Let

$$b = \frac{-\sum_{i=1, i\neq i'}^{n} (-1)^{i+j} a_{ij} \mathrm{Det}(M^{ij})}{(-1)^{i'+j} \mathrm{Det}(M^{i'j})}.$$

Replacing the entry $a_{i'j}$ by b, the new matrix A' has

$$\mathrm{Det}(A') = (-1)^{i'+j} b \mathrm{Det}(M^{i'j}) + \sum_{i=1, i \neq i'}^{n} (-1)^{i+j} a_{ij} \mathrm{Det}(M^{ij}) = 0$$

which shows that A' is noninvertible.

Prob. 3:

A matrix $\mathbf{A}$ is invertible if and only if its reduced echelon form is the identity matrix. This, in turn, is equivalent to the assertion that the rows (or columns) of $\mathbf{A}$ constitute a linearly independent set. Since the determinant of a matrix is 0 if and only if the rows (or columns) of the matrix form a linear dependent set, this is equivalent to the assertion that $\mathrm{Det}(A) \neq 0$.

Prob. 4:

Note that

$$\begin{bmatrix} 1 & 0 & 1 & 3 \\ -1 & 1 & -2 & -8 \\ 4 & 4 & 0 & -8 \\ -2 & -1 & 3 & 11 \\ 3 & 2 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -5 \\ 0 & 4 & -4 & -20 \\ 0 & -1 & 5 & 17 \\ 0 & 2 & -2 & -10 \end{bmatrix} \sim \begin{bmatrix} \boxed{1} & 0 & 1 & 3 \\ 0 & \boxed{1} & -1 & -5 \\ 0 & 0 & 4 & 12 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since the pivots are on the first, second and third columns, $\{(1,-1,4,-2,3),(0,1,4,-1,2),$ $(1,-2,0,3,1)\}$ is a basis for Span(S).

Prob. 5:

(a) With

$$[C\,|B\,]=\left[\begin{array}{cccc|cccc}1&0&0&0&1&1&-2&1\\0&1&0&0&0&0&1&-1\\0&0&1&0&-2&0&0&2\\0&0&0&1&1&-1&0&0\end{array}\right],$$

we have the transition matrix

$$[C\leftarrow B]=\begin{bmatrix}1&1&-2&1\\0&0&1&-1\\-2&0&0&2\\1&-1&0&0\end{bmatrix}.$$

(b) With

$$\left[\begin{array}{cccc|c}1&1&-2&1&-2\\0&0&1&-1&-1\\-2&0&0&2&1\\1&-1&0&0&1\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-2&1&-2\\0&0&1&-1&-1\\0&2&-4&4&-3\\0&-2&2&-1&3\end{array}\right]\to\left[\begin{array}{cccc|c}1&1&-2&1&-2\\0&1&-2&2&-\frac{3}{2}\\0&0&1&-1&-1\\0&0&-2&3&0\end{array}\right]\to$$

$$\left[\begin{array}{cccc|c}1&0&0&-1&-\frac{1}{2}\\0&1&-2&2&-\frac{3}{2}\\0&0&1&-1&-1\\0&0&0&1&-2\end{array}\right]\to\left[\begin{array}{cccc|c}1&0&0&0&-\frac{5}{2}\\0&1&0&0&-\frac{7}{2}\\0&0&1&0&-3\\0&0&0&1&-2\end{array}\right],$$

we have

$$[p(t)]_B=\begin{bmatrix}-\frac{5}{2}\\-\frac{7}{2}\\-3\\-2\end{bmatrix}.$$

Prob. 6:

Since

$$T((1,0,1))=A\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}-3\\-1\\1\end{bmatrix}=x_1\begin{bmatrix}1\\0\\1\end{bmatrix}+y_1\begin{bmatrix}2\\-1\\0\end{bmatrix}+z_1\begin{bmatrix}-2\\1\\-1\end{bmatrix},$$

$$T((2,-1,0))=A\begin{bmatrix}2\\-1\\0\end{bmatrix}=\begin{bmatrix}-5\\-5\\0\end{bmatrix}=x_2\begin{bmatrix}1\\0\\1\end{bmatrix}+y_2\begin{bmatrix}2\\-1\\0\end{bmatrix}+z_2\begin{bmatrix}-2\\1\\-1\end{bmatrix},$$

$$T((-2,1,-1)) = A\begin{bmatrix}-2\\1\\-1\end{bmatrix} = \begin{bmatrix}6\\5\\-1\end{bmatrix} = x_3\begin{bmatrix}1\\0\\1\end{bmatrix} + y_3\begin{bmatrix}2\\-1\\0\end{bmatrix} + z_3\begin{bmatrix}-2\\1\\-1\end{bmatrix},$$

we have a linear system

$$\begin{bmatrix}1 & 2 & -2\\0 & -1 & 1\\1 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1 & x_2 & x_3\\y_1 & y_2 & y_3\\z_1 & z_2 & z_3\end{bmatrix} = \begin{bmatrix}-3 & -5 & 6\\-1 & -5 & 5\\1 & 0 & -1\end{bmatrix}.$$

By solving the linear system,

$$\left[\begin{array}{ccc|ccc}1 & 2 & -2 & -3 & -5 & 6\\0 & -1 & 1 & -1 & -5 & 5\\1 & 0 & -1 & 1 & 0 & -1\end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & -1 & 1 & 0 & -1\\0 & 1 & -1 & 1 & 5 & -5\\0 & 0 & 1 & -6 & -15 & 17\end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc}1 & 0 & 0 & -5 & -15 & 16\\0 & 1 & 0 & -5 & -10 & 12\\0 & 0 & 1 & -6 & -15 & 17\end{array}\right],$$

we have the matrix representation of T relative to the basis B,

$$[T]_B = \begin{bmatrix}-5 & -15 & 16\\-5 & -10 & 12\\-6 & -15 & 17\end{bmatrix}.$$

Prob. 7:

(i) Since V contains the zero matrix, it is not an empty set.

(ii) For all $A_1, A_1 \in V$, $C(A_1 + A_2)B = CA_1B + CA_2B = O_{2\times2} + O_{2\times2} = O_{2\times2}$, so $A_1 + A_2 \in V$ and V is closed under the vector addition.

(iii) For all $A \in V$ and $\alpha \in \mathbb{R}$, $C(\alpha A)B = \alpha O_{2\times2} = O_{2\times2}$, so V is closed under the scalar multiplication.

From (i), (ii) and (iii), V is a subspace of $\mathcal{M}_{2\times2}$.

Let $A = \begin{bmatrix}a & b\\c & d\end{bmatrix}$, then $CAB = \begin{bmatrix}a-c & a-c\\c-a & c-a\end{bmatrix}$. Therefore $A \in V$ if and only of $a = c$ if and only if $A = \begin{bmatrix}a & b\\a & d\end{bmatrix} = a\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}$, where a, b and d are arbitrary scalars. Thus, $S = \left\{\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix}, \begin{bmatrix}0 & 1\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}\right\}$ spans V. Since the only solution to $a\begin{bmatrix}1 & 0\\1 & 0\end{bmatrix} + b\begin{bmatrix}0 & 1\\0 & 0\end{bmatrix} + d\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix} = O_{2\times2}$ is $a = 0, b = 0$ and $d = 0$, S is a linearly independent set.

Since S is a linearly independent subset of V and spans V, S is a basis for V and then $\mathrm{Dim}(V) = |S| = 3$.

Prob. 8:

(a) Yes. Since $\text{Det}(A-\lambda I) = \text{Det}((A-\lambda I)^T) = \text{Det}(A^T - \lambda I)$, A and A^T have the same characteristic polynomial. Therefore, A and A^T have the same eigenvalues.

(b) No. Suppose that 0 is an eigenvalue of A. Then we have $\text{Det}(A - 0I) = \text{Det}(A) = 0$, which implies that A is not invertible, a contradiction.

Prob. 9:

(a) We can directly use Theorem 1 in Section 4.2 of the textbook to obtain

$$\text{Det}(A - \lambda I_{3\times 3}) = \text{Det}(\begin{bmatrix} -1-\lambda & 0 & 4 \\ 2 & -2-\lambda & -10 \\ -1 & 0 & 3-\lambda \end{bmatrix}) = (-2-\lambda)\ \text{Det}(\begin{bmatrix} -1-\lambda & 4 \\ -1 & 3-\lambda \end{bmatrix})$$

$$= (-\lambda - 2) \times [(-\lambda-1)(-\lambda+3)+4] = (-\lambda-2)\times(\lambda-1)^2.$$

Thus the eigenvalues of T are -2 and 1.

When $\lambda = -2$:

$$\left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 2 & 0 & -10 & 0 \\ -1 & 0 & 5 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 0 & -18 & 0 \\ 0 & 0 & 9 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 & = 0 \\ x_3 & = 0 \end{cases}$$

The eigenspace corresponding to the eigenvalue -2 is the vector subspace spanned by $\mathbf{v}_1 = (0, 1, 0)$.

When $\lambda = 1$:

$$\left[\begin{array}{ccc|c} -2 & 0 & 4 & 0 \\ 2 & -3 & -10 & 0 \\ -1 & 0 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & -3 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \begin{cases} x_1 - 2x_3 & = & 0 \\ x_2 + 2x_3 & = & 0 \end{cases}$$

The eigenspace corresponding to the eigenvalue 1 is the vector subspace spanned by $\mathbf{v}_2 = (2, -2, 1)$.

(b) The algebraic multiplicity and the geometric multiplicity of the eigenvalue -2 are 1 and 1 respectively. While the algebraic multiplicity and the geometric multiplicity of the eigenvalue 1 are 2 and 1 respectively.

(c) Since the sum of the geometric multiplicities of the two eigenvectors -2 and 1 is 2 which is less than the dimension of $\mathbb{R}^3$, we are unable to find an eigenbasis B for $\mathbb{R}^3$ so that the matrix representation $[T]_B$ of T relative to B is diagonal.

Prob. 10:

Since A is similar to a diagonal matrix D $= \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, there is an invertible matrix P such that $A = PDP^{-1}$. Since

$$A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1} \text{ and } I_{nxn} = PI_{nxn}P^{-1},$$

we have

$$A^2 + 2A - 15I_{nxn} = P(D^2 + 2D - 15I_{nxn})P^{-1} = 0_{nxn}$$

so that

$$0_{nxn} = D^2 + 2D - 15I_{nxn} = \begin{bmatrix} \lambda_1^2 + 2\lambda_1 - 15 & 0 & \cdots & 0 \\ 0 & \lambda_2^2 + 2\lambda_2 - 15 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n^2 + 2\lambda_n - 15 \end{bmatrix}$$

which shows that all eigenvalues satisfy the quadratic equation $\lambda^2 + 2\lambda - 15 = (\lambda + 5)(\lambda - 3) = 0$. The eigenvalues can only be -5 or 3.