EE205003 Linear Algebra, 2020 Fall Semester

Midterm I

DATE: Nov. 4th, 2020

1. (5%) Let A_n be a subset of a universal set S for each positive integer n. Let x be an element of S. What is the negative (denial) of the assertion that there is a positive integer n such that for all positive integers $m \geq n$, $x \in A_m$.

2. (5%) Under what conditions does the augmented matrix correspond a consistent linear system: $\left[\begin{array}{ccc|c} 3 & 0 & -1 & a \\ -2 & -1 & 3 & b \\ -1 & -2 & 7 & c \\ 1 & -1 & 2 & d \end{array}\right]$?

3. (10%) A plane H in $\mathbf{R}^5$ is represented parametrically by the equation: $\mathbf{x} = \mathbf{w} + s\mathbf{x} + t\mathbf{y} + u\mathbf{z}$, where $\mathbf{w} = (-1, 2, 0, -1, 0), \mathbf{x} = (1, 0, 1, -2, -1), \mathbf{y} = (1, 0, -1, 0, 2), \mathbf{z} = (3, 0, 1, -1, 1)$ and $s, t, u \in \mathbb{R}$. Please represent this plane by a linear system $A\mathbf{x} = \mathbf{b}$, i.e., H is the solution set of $A\mathbf{x} = \mathbf{b}$.

4. True or false. Please give reason(s) such as a proof or a counterexample, otherwise no credits.

   (a) (5%) If T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$, then T cannot be surjective.

   (b) (5%) There is a 2×3 matrix A and a 3×2 matrix B such that $AB = I_{2\times 2}$.

   (c) (5%) If A, B are two 2×2 matrices such that $AB = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, then $AB = BA$.

   (d) (5%) The set of all matrices in $\mathbb{R}^{n\times n}$ which commute with a fixed matrix A in $\mathbb{R}^{n\times n}$ form a vector subspace of $\mathbb{R}^{n\times n}$.

   (e) (5%) The standard monomial functions $p_0(t) = 1, p_1(t) = t, p_2(t) = t^2, p_3(t) = t^3$ form a linearly independent set if we take the domain of these functions to be the set $A = \{-1, 0, 1\}$.

5. (5%) Let

$$A = \begin{bmatrix} -1 & b & 1 \\ d & e & f \end{bmatrix}.$$

Find values of b, d, e, f so that the linear transformation $\mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x}$ maps $\mathbb{R}^3$ onto $\mathbb{R}^2$.

6. Consider a linear transformation $T(\mathbf{x}) = A\mathbf{x}$ with $A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & -6 \\ -1 & 3 & 1 \\ 0 & -2 & 2 \end{bmatrix}$.

   (a) (2%) Please find the domain and co-domain of T.

   (b) (4%) Please determine whether T is injective.

   (c) (4%) Please determine whether T is surjective.

7. (10%) Let V be the set of all ordered pairs (x, y) of real numbers. Define a vector addition

$$(x, y) \oplus (x', y') \triangleq (x + x', 0)$$

and a scalar multiplication

$$\alpha \odot (x, y) \triangleq (\alpha x, 0) \oplus (\alpha x', 0) = (\alpha(x$$

for all $x, x', y, y', \alpha \in \mathbb{R}$. Is $(V, \oplus, \odot)$ a vector space over $\mathbb{R}$? Please give reason(s), otherwise no credits.

8. Consider a matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$.

   (a) (5%) Find all right inverses of A.

   (b) (5%) Find all left inverse of A.

9. (10%) Assume that an $m \times n$ matrix A has the property that AA^T has a left inverse. Please show that the rows of A form a linearly independent set.

10. (10%) Please show that an $m \times n$ matrix A has a left inverse if and only if $\operatorname{rank}(A) = n$.