EECS205003 Linear Algebra, Fall 2020

Midterm # 1, Solutions

Prob. 1:

For all positive integers n, there exists a positive integer $m \geq n$ such that $x \notin A_m$.

Prob. 2:

$$\left[\begin{array}{ccc|c} 3 & 0 & -1 & a \\ -2 & -1 & 3 & b \\ -1 & -2 & 7 & c \\ 1 & -1 & 2 & d \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -1 & 2 & d \\ 0 & 3 & -7 & a-3d \\ 0 & -3 & 7 & b+2d \\ 0 & -3 & 9 & c+d \end{array}\right] \sim$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & -\frac{c}{3}+\frac{2d}{3} \\ 0 & 1 & -3 & -\frac{c}{3}-\frac{d}{3} \\ 0 & 0 & 2 & a+c-2d \\ 0 & 0 & -2 & b-c+d \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{a}{2}+\frac{c}{6}-\frac{d}{3} \\ 0 & 1 & 0 & \frac{3a}{2}+\frac{7c}{6}-\frac{10d}{3} \\ 0 & 0 & 1 & \frac{a}{2}+\frac{c}{2}-d \\ 0 & 0 & 0 & a+b-d \end{array}\right],$$

which shows that when $a+b-d=0$, the augmented matrix corresponds a consistent linear system.

Prob. 3:

A point $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5) \in \mathbb{R}^5$ is in the 3-dimensional affine space H if and only if there exist $s, t, u \in \mathbb{R}$ such that $\mathbf{x} - \mathbf{w} = s\mathbf{p} + t\mathbf{q} + u\mathbf{r}$, i.e., the following linear system is consistent:

$$\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \\ 1 & -1 & 1 \\ -2 & 0 & -1 \\ -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} s \\ t \\ u \end{bmatrix} = \begin{bmatrix} x_1+1 \\ x_2-2 \\ x_3 \\ x_4+1 \\ x_5 \end{bmatrix} \Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1+1 \\ 0 & 0 & 0 & x_2-2 \\ 1 & -1 & 1 & x_3 \\ -2 & 0 & -1 & x_4+1 \\ -1 & 2 & 1 & x_5 \end{array}\right] \text{ is consistent}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1+1 \\ 0 & -2 & -2 & -x_1+x_3-1 \\ 0 & 2 & 5 & 2x_1+x_4+3 \\ 0 & 3 & 4 & x_1+x_5+1 \\ 0 & 0 & 0 & x_2-2 \end{array}\right] \text{ is consistent}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & x_1+1 \\ 0 & 1 & 1 & \frac{1}{2}x_1-\frac{1}{2}x_3+\frac{1}{2} \\ 0 & 0 & 3 & x_1+x_3+x_4+2 \\ 0 & 3 & 4 & x_1+x_5+1 \\ 0 & 0 & 0 & x_2-2 \end{array}\right] \text{ is consistent.}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & -\frac{1}{6}x_1-\frac{1}{6}x_3-\frac{2}{3}x_4-\frac{5}{6} \\ 0 & 1 & 0 & \frac{1}{6}x_1-\frac{5}{6}x_3-\frac{1}{3}x_4-\frac{1}{6} \\ 0 & 0 & 1 & \frac{1}{3}x_1+\frac{1}{3}x_3+\frac{1}{3}x_4+\frac{2}{3} \\ 0 & 0 & 0 & -\frac{5}{6}x_1+\frac{7}{6}x_3-\frac{1}{3}x_4+x_5-\frac{7}{6} \\ 0 & 0 & 0 & x_2-2 \end{array}\right] \text{ is consistent.}$$

Therefore a point $\mathbf{x} = (x_1, x_2, x_3, x_4, x_5)$ is in the 3-dimensional affine space H if and only if $\begin{cases} -\frac{5}{6}x_1 + \frac{7}{6}x_3 - \frac{1}{3}x_4 + x_5 = \frac{7}{6} \\ x_2 = 2 \end{cases}$ if and only if

$$\begin{bmatrix} -\frac{5}{6} & 0 & \frac{7}{6} & -\frac{1}{3} & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -\frac{7}{6} \\ 2 \end{bmatrix}.$$

Prob. 4:

1. True. If T is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$, then $\mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x}$, where A is a matrix in $\mathbb{R}^{4\times3}$. Therefore, the reduced row echelon form of A will have at least one row which has no pivot. By Theorem 1.2.4, we can conclude that A is not onto.

2. True. For example, let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then we have $AB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

3. False. For example, let $A = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$. Then we have $AB = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = BA$.

4. True. To check whether a subset of a vector space is a subspace or not, we only need to check the closure property under vector addition and scalar multiplication, the existence of a zero vector, and the existence of an additive inverse.

   (a) (Closure under additon) For all B and C that commute with A, $A(B+C) = AB + AC = BA + CA = (B+C)A$, which shows that $B + C$ is also commute with A.

   (b) (Closure under scalar multiplication) For all $\alpha \in \mathbb{R}$ and C commute with A, $A(\alpha C) = \alpha AC = \alpha CA = (\alpha C)A$, which shows that αC is also commute with A.

   (c) (Existence of a zero vector) Since $O + A = A = A + O$, the zero matrix O commutes wit A.

   (d) (Existence of an additive inverse) For all D that commute with A, $A(-D) = -(AD) = -(DA) = (-D)A$, which shows that the additive inverse $-D$ of D is also commute with A.

5. False. Consider an arbitrary linear relation in the set $\{p_0(t), p_1(t), p_2(t), p_3(t)\}$,

$$a_0p_0(t) + a_1p_1(t) + a_2p_2(t) + a_3p_3(t) = a_0 + a_1t + a_2t^2 + a_3t^3 = 0,\ \forall\, t \in A = \{-1, 0, 1\},$$

where a_0, a_1, a_2, a_3 are real numbers. Since the domain of these functions is $A = \{$ -1, 0, 1 $\}$, the linear relation of the four functions is equivalent to its evaluation at all points $t \in A$, i.e.,

$$\begin{aligned} a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 &= 0 \\ a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 &= 0 \\ a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 &= 0. \end{aligned}$$

To solve the homogeneous linear system in above, we consider the coefficient matrix

$$\begin{bmatrix} 1 & -1 & 1 & -1 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix},$$

which has at most three pivot positions and the variables corresponding to columns without a pivot are free variables so that the homogeneous linear system has a nontrivial solution. Thus the set $\{p_0(t), p_1(t), p_2(t), p_3(t)\}$ is linearly dependent.

Prob. 5:

$$\begin{bmatrix} -1 & b & 1 \\ d & e & f \end{bmatrix} \sim \begin{bmatrix} 1 & -b & -1 \\ 0 & e+bd & f+d \end{bmatrix}$$

By **Theorem 1.2.4**, we know that if the linear transformation is onto, then there is a pivot in every row of the reduced row echelon form of A. Therefore, if $e + bd \neq 0$ or $f + d \neq 0$, then the linear transformation $\mathbf{x} \mapsto T(\mathbf{x}) = A\mathbf{x}$ will map $\mathbb{R}^3$ onto $\mathbb{R}^2$.

Prob. 6:

(a) Domain: $\mathbb{R}^3$. Co-domain: $\mathbb{R}^4$.

(b) Since $\begin{bmatrix} 1 & -1 & 0 \\ 2 & 0 & -6 \\ -1 & 3 & 1 \\ 0 & -2 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $Ker(A) = \{0\}$ and T is injective.

(c) Since the reduced row echelon form of A has a zero row, there exists $\mathbf{b} \in \mathbb{R}^4$ such that $A\mathbf{x} = \mathbf{b}$ is inconsistent by **Theorem 1.2.4**. Therefore T is not surjective.

Prob. 7:

No, since there is no zero vector for $(V, \oplus, \odot)$. Suppose $\mathbf{0} = (a, b)$ is a zero vector of V. For all $x, y \in \mathbb{R}$ such that $y \neq 0$, $(x, y) \oplus (a, b) = (x + a, 0) \neq (x, y)$. Therefore $(V, \oplus, \odot)$ has no zero vector and it is not a vector space over $\mathbb{R}$.

Prob. 8:

1.

$$\left[\begin{array}{cc|ccc} 1 & -1 & 1 & 0 & 0 \\ -1 & 2 & 0 & 1 & 0 \\ 3 & 1 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 4 & -3 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -7 & -4 & 1 \end{array}\right],$$

which is inconsistent so that A has no right inverse.

2. Suppose C is a left inverse of A, then $CA = I_2$ and $A^T C^T = I_2^T = I_2$ such that

$$\left[\begin{array}{ccc|cc} 1 & -1 & 3 & 1 & 0 \\ -1 & 2 & 1 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & -1 & 3 & 1 & 0 \\ 0 & 1 & 4 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|cc} 1 & 0 & 7 & 2 & 1 \\ 0 & 1 & 4 & 1 & 1 \end{array}\right]$$

$$\Rightarrow \begin{cases} c_{11} + 7c_{13} = 2 \\ c_{12} + 4c_{13} = 1 \\ c_{21} + 7c_{23} = 1 \\ c_{22} + 4c_{23} = 1 \end{cases} \Rightarrow \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c_{13} \begin{bmatrix} -7 \\ -4 \\ 1 \end{bmatrix} \text{ and } \begin{bmatrix} c_{21} \\ c_{22} \\ c_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_{23} \begin{bmatrix} -7 \\ -4 \\ 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_{11} & c_{12} & c_{13} \\ c_{21} & c_{22} & c_{23} \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} + s \begin{bmatrix} -7 & -4 & 1 \\ 0 & 0 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & 0 & 0 \\ -7 & -4 & 1 \end{bmatrix}$$

are all left inverses of A, where s, t are real numbers.

Prob. 9:

Let B be a left inverse of AA^T, i.e., $B(AA^T) = I$. Since AA^T is a square matrix, B is also a right inverse of AA^T by **Theorem 3.2.4**, i.e., $(AA^T)B = I_{m\times m} \Rightarrow A(A^T B) = I_{m\times m}$ and A has a right inverse. By **Theorem 3.2.B**, $\text{Col}(A) = \mathbb{R}^m$, i.e., the linear system $A\mathbf{x} = \mathbf{b}$ is consistent for every $\mathbf{b} \in \mathbb{R}^m$. By **Theorem 1.2.4**, each row of A has a pivot position. And by **Theorem 1.3.11**, the rows of A form a linearly independent set.

Prob. 10:

An $m \times n$ matrix A has a left inverse if and only if $\text{Ker}(A) = \{\mathbf{0}\}$ by **Theorem 3.2.A** if and only if every column in the reduced row echelon form of A has a pivot if and only if $\text{Rank}(A) = n$.