2018 Fall EECS205003 Linear Algebra - Quiz 4 Solution

Name: ID:

1. Given $\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\2\\0\\1\end{bmatrix}$ spanning nullspace $\mathbf{N}(A)$ and $\begin{bmatrix}1\\-1\\2\end{bmatrix}$ and $\begin{bmatrix}-1\\0\\-1\end{bmatrix}$ spanning column space $\mathbf{C}(A)$ of matrix $A_{3\times 5}$, please try to recover A.

   (a) Try to recover reduced row echelon form R from nullspace.

   According to nullspace, we can know there are only 2 pivots. In addition, two pivots are at col. 1 and col. 3, respectively. Let $\mathbf{r}_i$ be the i_{th} col. of R.

$$\mathbf{r}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}, \mathbf{r}_3 = \begin{bmatrix}0\\1\\0\end{bmatrix}, 2\mathbf{r}_1 + \mathbf{r}_2 = \mathbf{0}, \mathbf{r}_1 - \mathbf{r}_3 + \mathbf{r}_4 = \mathbf{0}, -\mathbf{r}_1 + 2\mathbf{r}_3 + \mathbf{r}_5 = \mathbf{0}$$

   Thus we can recover

$$R = \begin{bmatrix}1 & -2 & 0 & -1 & 1\\ 0 & 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

   (b) Since we know the steps to get R by $A = LU \to U \to R$, try to reverse the process and retrieve A. You only need to answer one possible A.

   We obtain R from U by rescale pivots to 1 and eliminate other elements in pivot columns to 0. Then with non-zero n_1 and $n_2 \in \mathbb{R}$ and $c \in \mathbb{R}$, we can reconstruct U s.t. $A = LU$ in a bottom manner.

$$R = \begin{bmatrix}1 & -2 & 0 & -1 & 1\\ 0 & 0 & 1 & 1 & -2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}1 & -2 & 0 & -1 & 1\\ 0 & 0 & n_2 & n_2 & -2n_2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$\to \begin{bmatrix}1 & -2 & cn_2 & -1+cn_2 & 1-2cn_2\\ 0 & 0 & n_2 & n_2 & -2n_2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} \to \begin{bmatrix}n_1 & -2n_1 & cn_1n_2 & -n_1+cn_1n_2 & n_1-2cn_1n_2\\ 0 & 0 & n_2 & n_2 & -2n_2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix} = U$$

$$\Rightarrow A = LU = \begin{bmatrix}1 & 0 & 0\\ l_{21} & 1 & 0\\ l_{31} & l_{32} & 1\end{bmatrix}\begin{bmatrix}n_1 & -2n_1 & cn_1n_2 & -n_1+cn_1n_2 & n_1-2cn_1n_2\\ 0 & 0 & n_2 & n_2 & -2n_2\\ 0 & 0 & 0 & 0 & 0\end{bmatrix}$$

$$= \begin{bmatrix}n_1 & -2n_1 & cn_1n_2 & -n_1+cn_1n_2 & n_1-2cn_1n_2\\ l_{21}n_1 & -2l_{21}n_1 & cl_{21}n_1n_2+n_2 & l_{21}(-n_1+cn_1n_2)+n_2 & l_{21}(n_1-2cn_1n_2)-2n_2\\ l_{31}n_1 & -2l_{31}n_1 & l_{31}cn_1n_2+l_{32}n_2 & l_{31}(-n_1+cn_1n_2)+l_{32}n_2 & l_{31}(n_1-2cn_1n_2)-2l_{32}n_2\end{bmatrix}$$

   If substituting vectors spanning column space into pivot columns of A

$$\begin{bmatrix}n_1\\ l_{21}n_1\\ l_{31}n_1\end{bmatrix} = \begin{bmatrix}1\\-1\\2\end{bmatrix}, \begin{bmatrix}cn_1n_2\\ l_{21}cn_1n_2+n_2\\ l_{31}cn_1n_2+l_{32}n_2\end{bmatrix} = \begin{bmatrix}-1\\0\\-1\end{bmatrix}$$

   we have $n_1 = 1, l_{21} = -1, l_{31} = 2, c = 1, n_2 = -1$ and $l_{32} = -1$

$$\Rightarrow A = \begin{bmatrix}1 & -2 & -1 & -2 & 3\\ -1 & 2 & 0 & 1 & -1\\ 2 & -4 & -1 & -3 & 4\end{bmatrix}$$

If we substitute them in opposite manner

$$\begin{bmatrix} n_1 \\ l_{21}n_1 \\ l_{31}n_1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} cn_1n_2 \\ l_{21}cn_1n_2 + n_2 \\ l_{31}cn_1n_2 + l_{32}n_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

we have $n_1 = -1, l_{21} = 0, l_{31} = 1, c = 1, n_2 = -1$ and $l_{32} = -1$

$$\Rightarrow A = \begin{bmatrix} -1 & 2 & 1 & 2 & -3 \\ 0 & 0 & -1 & -1 & 2 \\ -1 & 2 & 2 & 3 & -5 \end{bmatrix}$$