2018 Fall EECS205003 Linear Algebra - Homework 3 solutions

Name: ID:

1. $A^TCAx = \begin{bmatrix} 4 & -2 & -2 & 0 \\ -2 & 8 & -3 & -3 \\ -2 & -3 & 8 & -3 \\ 0 & -3 & -3 & 6 \end{bmatrix} x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ -1 \end{bmatrix}$ gives potentials

   $x = (\frac{5}{12}, \frac{1}{6}, \frac{1}{6}, 0)$ (grounded $x_4 = 0$ and solved 3 equations);

   $y = -CAx = (\frac{2}{3}, \frac{2}{3}, 0, \frac{1}{2}, \frac{1}{2})$.

2. (a) Find the condition on (b_1, b_2, b_3) for $A\mathbf{x} = \mathbf{b}$ to be solvable.

   $$A = \begin{bmatrix} 1 & 3 & 1 & 2 \\ 2 & 6 & 4 & 8 \\ 0 & 0 & 2 & 4 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

   $[\ A\ \mathbf{b}\ ] \to [\ R\ \mathbf{d}\ ]$

   $$\begin{bmatrix} 1 & 3 & 1 & 2 & b_1 \\ 2 & 6 & 4 & 8 & b_2 \\ 0 & 0 & 2 & 4 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 0 & 0 & b_1 - 0.5 * (b_2 - b_1) \\ 0 & 0 & 1 & 2 & 0.5 * (b_2 - 2b_1) \\ 0 & 0 & 0 & 0 & b_3 - (b_2 - 2b_1) \end{bmatrix}$$

   If $A\mathbf{x} = \mathbf{b}$ is solvable, then the third row should be $0 = 0$. Therefore, $b_3 - b_2 + 2b_1 = 0$

   (b) If $\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, find the complete solution.

   $$[\ R\ \mathbf{d}\ ] = \begin{bmatrix} 1 & 3 & 0 & 0 & 0.5 \\ 0 & 0 & 1 & 2 & 0.5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, R\mathbf{x_p} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0 \end{bmatrix} \Longrightarrow \mathbf{x_p} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix}$$

   $$R\mathbf{x_n} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \mathbf{x} = \mathbf{x_p} + \mathbf{x_n} = \begin{bmatrix} 0.5 \\ 0 \\ 0.5 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

3. Elimination on the matrices formed from these column vectors gives for (a):

   $$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 3 \\ 2 & 3 & 5 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -6 \\ 0 & -1 & -1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -6 \\ 0 & 0 & \frac{1}{2} \end{bmatrix}$$

   So this matrix has rank 3 and the columns are independent

   For (b) we get

   $$\begin{bmatrix} 1 & 2 & -3 \\ -3 & 1 & 2 \\ 2 & -3 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & -7 & 7 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & -3 \\ 0 & 7 & -7 \\ 0 & 0 & 0 \end{bmatrix}$$

so this matrix has rank 2 and the columns are dependent.

4. Let $A = \begin{bmatrix}1&0&0\\3&1&0\\1&2&1\end{bmatrix}\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\end{bmatrix}$. Find bases of four subspaces without computing A.

   From $\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\end{bmatrix}$ we can know basis is on $1_{st}, 2_{nd}$ and 3_{rd} columns. As $A = \begin{bmatrix}1&2&3&4\\3&7&11&15\\1&4&8&12\end{bmatrix}$, $\mathbf{C}(A)$ is then spanned by $\begin{bmatrix}1\\3\\1\end{bmatrix}$, $\begin{bmatrix}2\\7\\4\end{bmatrix}$ and $\begin{bmatrix}3\\11\\8\end{bmatrix}$.

   The reduce matrix $\begin{bmatrix}1&2&3&4\\0&1&2&3\\0&0&1&2\end{bmatrix} \to \begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&1&2\end{bmatrix} \to \begin{bmatrix}1&0&0&0\\0&1&0&-1\\0&0&1&2\end{bmatrix} = R$

   Thus the only basis of $\mathbf{N}(A)$ is $\begin{bmatrix}0\\1\\-2\\1\end{bmatrix}$ and bases of row space $\mathbf{C}(A^{\mathsf{T}}) = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$, $\begin{bmatrix}0\\1\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\0\\1\\2\end{bmatrix}$ and left nullspace $\mathbf{N}(A^{\mathsf{T}})$ has no basis.

5. Assume two $m \times n$ matrices $A = \begin{bmatrix}I & F_A\\ 0 & 0\end{bmatrix}$ and $B = \begin{bmatrix}I & F_B\\ 0 & 0\end{bmatrix}$ has the same four subspaces. Proof $F_A = F_B$.

   As A and B, they have the same number of pivot columns and, in other word, the same number of free columns. If $F_A \neq F_B$, A and B have different solution and thus have different nullspace(contradiction). Thus $F_A = F_B$.

6. (a) A has rank 2, with the nullspace having basis as $\begin{bmatrix}1\\1\\1\end{bmatrix}$. So X will looked like:

   $$\begin{bmatrix}a&b&c\\a&b&c\\a&b&c\end{bmatrix}$$

   And the dimension is 1.

   (b) The columns of the form AX are linear combinations of the columns of A . And its columns should sum to 0.

   $$\begin{bmatrix}a&b&c\\d&e&f\\-a-d&-b-e&-c-f\end{bmatrix}$$

   And the dimension is 2.