2018 Fall EECS205003 Linear Algebra - Midterm 2 Solution

1. For $A = \begin{bmatrix} 2 & 4 & 3 \\ 2 & 5 & 7 \\ 4 & 9 & 10 \end{bmatrix}$

   (a) The first column and the second column are independent, but the third column is dependent with the first and the second.

   (b) The space is $N(A^T)$, and the basis is $\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix}$

   (c) The space is $N(A)$, and the basis is $\begin{bmatrix} 13 \\ -8 \\ 2 \end{bmatrix}$

2. (a) The vectors $(-1,1,0)^T$ and $(-1,0,1)^T$ form a basis for the subspace $x+y+z=0$. Let A be the matrix whose columns are the two vectors found above. Thus the projection matrix P onto the subspace $x+y+z=0$ is

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} 1/3 \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^{-1} \begin{bmatrix} -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} -1 & -1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

$$= 1/3 \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$$

   The projection of $(1,2,6)^T$ onto the plane $x+y+z=0$ is thus simply

$$p = P \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 3 \end{bmatrix}$$

   (b) To find the line y = mx+b which best the four data points $(1,2), (2,1), (3,3)$ and $(4,2)$ in the sense of least squares.

$$\begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

   We can find the least squares solution by multiplying by the transpose of the matrix

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ 2 \end{bmatrix}$$

   which becomes $\begin{bmatrix} 30 & 10 \\ 10 & 4 \end{bmatrix} \begin{bmatrix} m \\ b \end{bmatrix} = \begin{bmatrix} 21 \\ 8 \end{bmatrix}$

Solving this by elimination on an augmented matrix we get

$$\left[\begin{array}{ccc} 30 & 10 & 21 \\ 10 & 4 & 8 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 0 & -2 & -3 \\ 5 & 2 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc} 0 & 2 & 3 \\ 5 & 0 & 1 \end{array}\right]$$

Therefore $2b = 3$ and $5m = 1$, so $m = 1/5$ and $b = 3/2$ and the best line is $y = x/5 + 3/2$.

(c) Since $v_1.v_2 = 1$ the vectors $v_1 and v_2$ are linearly independent. S is not a orthogonal basis.

We apply the Gram-Schmidt process to generate an orthogonal basis as follows.

We define vectors u_1, u_2 by the following formula. Then B=u_1, u_2 is an orthogonal basis of Span(S).

5cm$u_1 = v_1$
5cm$u_2 = v_2 - \frac{u_1.v_2}{u_1.u_1}u_1$

2.30cmSince we have $u_1.u_1 = v_1.v_2 = 2$ and $v_1.v_2 = u_1.u_2 = 1$

We compute 3cm $u_2 = v_2 - \frac{1}{2}u_1$

5cm$= \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

5cm$= \begin{bmatrix} -1/2 \\ 1 \\ 1/2 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}.$

Therefore the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{2}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}$ is the orthogonal basis of Span(S).

(d) Use the Gram - Schmidt process to find $\left\{ \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ which is an

orthogonal basis for col A = span $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix} \right\}$. So Q = $\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}$.

Since U has orthogonal columns, $Q^TQ = I$. So if $A = QR$, then

$$R = Q^TA = \begin{bmatrix} \frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 2 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} \sqrt{2} & 2\sqrt{2} \\ 0 & 3 \end{bmatrix}$$

3. (a) $U^TU = 4I \Rightarrow U^{-1} = \frac{1}{4}U^T = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix}$

(b) NO, because the any $< u_i, u_i > \neq 1$

(c) $c_3 = \frac{< y, u_3 >}{\|u_3\|^2} = 0$, $c_4 = \frac{< y, u_4 >}{\|u_4\|^2} = \frac{4}{4} = 1$

(d) $Proj, w(y) = \frac{< y, u_1 >}{\|u_1\|^2}u_1 + \frac{< y, u_2 >}{\|u_2\|^2}u_2 + \frac{< y, u_3 >}{\|u_3\|^2}u_3$

$$= \frac{10}{4}\begin{bmatrix}1\\1\\1\\1\end{bmatrix} + \frac{-2}{4}\begin{bmatrix}1\\-1\\1\\-1\end{bmatrix} + \frac{0}{4}\begin{bmatrix}-1\\1\\1\\-1\end{bmatrix} = \begin{bmatrix}2\\3\\2\\3\end{bmatrix}$$

4. (a) $det(A) = (2a)det\begin{bmatrix}a+d & 0 & b\\0 & a+d & c\\c & b & 2d\end{bmatrix} - (c)det\begin{bmatrix}b & 0 & b\\c & a+d & c\\0 & b & 2d\end{bmatrix} + (b)det\begin{bmatrix}b & a+d & b\\c & 0 & c\\0 & c & 2d\end{bmatrix}$

$= 2a[2d(a+d)^2 - 2bc(a+d)] - c[2bd(a+d)] + b[-2dc(a+d)]$

$= 4ad(a+d)^2 - 4abc(a+d) - 4bcd(a+d)$

$= 4(a+d)^2(ad-bc)$

(b) If $a+d=0$ or $ad-bc=0$, then $det(A)=0$. A has no inverse.

(c) Since $det(AB) = det(A)det(B)$, and $det(B)$ is equal to

$$det(B) = det\begin{bmatrix}1 & 101 & 201 & 301\\2 & 102 & 202 & 302\\3 & 103 & 203 & 303\\4 & 104 & 204 & 304\end{bmatrix} = det\begin{bmatrix}1 & 100 & 201 & 301\\2 & 100 & 202 & 302\\3 & 100 & 203 & 303\\4 & 100 & 204 & 304\end{bmatrix} = det\begin{bmatrix}1 & 100 & 1 & 301\\2 & 100 & 2 & 302\\3 & 100 & 3 & 303\\4 & 100 & 4 & 304\end{bmatrix}$$

column 1 and column 3 are the same, so $det(B) = 0 \Rightarrow det(AB) = 0$

5. (a) $\det(M) = (-1)\det\begin{bmatrix}a & b & 0 & 0\\c & d & 0 & 0\\0 & 0 & a & d\\0 & 0 & c & d\end{bmatrix} = (-1)(ad-bc)^2$

(b) $A = R_{23}M = \begin{bmatrix}a & b & 0 & 0\\c & d & 0 & 0\\0 & 0 & a & d\\0 & 0 & c & d\end{bmatrix}$

$A^{-1} = M^{-1}R_{23}^{-1} \rightarrow M^{-1} = A^{-1}C_{23}$

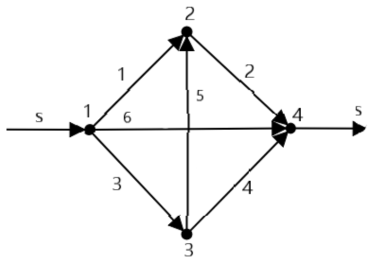
$$= \frac{1}{ad-bc}\begin{bmatrix}d & -b & 0 & 0\\-c & a & 0 & 0\\0 & 0 & d & -b\\0 & 0 & -c & a\end{bmatrix}\begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

$$= \frac{1}{ad-bc}\begin{bmatrix}d & 0 & -b & 0\\-c & 0 & a & 0\\0 & d & 0 & -b\\0 & -c & 0 & a\end{bmatrix}$$

6. The following figure shows electrical network **G**. The conductance $(c_1, c_2, c_3, c_4, c_5, c_6) = (\frac{1}{a}, \frac{1}{2a}, \frac{1}{a}, \frac{1}{2a}, \frac{1}{a}, \frac{1}{4a})$ to corresponding edges for some constant $a > 0$. And the current s flows into node 1 and flows out of node 4. Please answer the following questions.

(a) Represent the incident matrix A and try to use loops in **G** to indicate basis of left nullspace of A instead of computing elimination.

$$A = \begin{bmatrix}-1 & 1 & 0 & 0\\0 & -1 & 0 & 1\\-1 & 0 & 1 & 0\\0 & 0 & -1 & 1\\0 & 1 & -1 & 0\\-1 & 0 & 0 & 1\end{bmatrix}$$



Let x_i be the potential at node i and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \end{bmatrix}^\intercal$ denote the potential vector.

From $A\mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} x_2 - x_1 \\ x_4 - x_2 \\ x_3 - x_1 \\ x_4 - x_3 \\ x_2 - x_3 \\ x_4 - x_1 \end{bmatrix} \Rightarrow x_1 = x_2 = x_3 = x_4$, basis of $\mathbf{N}(A)$ is $c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$\text{rank(A)} = 4 - 1 = 3 \Rightarrow \dim(\mathbf{N}(A^\intercal)) = 6 - 3 = 3$

Let $\mathbf{y}$ denote the loop where $y_i \in \{-1, 1\}$ representing direction of loop.

According to figure, we can obtain all loops

$$\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ -1 \\ -1 \end{bmatrix}$$

and select 3 linear independent $\mathbf{y}$ as basis of $\mathbf{N}(A^\intercal)$, saying

$$\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

(b) Suppose that potential at node 1 is v. Analyse potential at each node and current on each edge.

Via Kirchhoff's current law and Ohm's law, we can obtain the equation

$$A^T C A = \begin{bmatrix} c_1 + c_3 + c_6 & -c_1 & -c_3 & -c_6 \\ -c_1 & c_1 + c_2 + c_5 & -c_5 & -c_2 \\ -c_3 & -c_5 & c_3 + c_4 + c_5 & -c_4 \\ -c_6 & -c_2 & -c_4 & c_2 + c_4 + c_6 \end{bmatrix} = \frac{1}{a} \begin{bmatrix} \frac{9}{4} & -1 & -1 & \frac{-1}{4} \\ -1 & \frac{5}{2} & -1 & \frac{-1}{2} \\ -1 & -1 & \frac{5}{2} & \frac{-1}{2} \\ \frac{-1}{4} & \frac{-1}{2} & \frac{-1}{2} & \frac{5}{4} \end{bmatrix}$$

$$\Rightarrow A^T C A\mathbf{x} = \frac{1}{a} \begin{bmatrix} \frac{9}{4}x_1 - x_2 - x_3 - \frac{1}{4}x_4 \\ -x_1 + \frac{5}{2}x_2 - x_3 - \frac{1}{2}x_4 \\ -x_1 - x_2 + \frac{5}{2}x_3 - \frac{1}{2}x_4 \\ \frac{-1}{4}x_1 - \frac{1}{2}x_2 - \frac{1}{2}x_3 + \frac{5}{4}x_4 \end{bmatrix} = \begin{bmatrix} s \\ 0 \\ 0 \\ -s \end{bmatrix}$$

$$\Rightarrow \begin{array}{rrrl} 4x_2 & +4x_3 & +x_4 & = 9v - 4as \\ 5x_2 & -2x_3 & -x_4 & = 2v \\ -2x_2 & +5x_3 & -x_4 & = 2v \\ -2x_2 & -2x_3 & +5x_4 & = v - 4as \end{array} \quad \Rightarrow \quad \begin{array}{rl} x_1 & = v \\ x_2 = x_3 & = v - \frac{4}{11}as \\ x_4 & = v - \frac{12}{11}as \end{array}$$

And current on each edge is

$$-CAx = -\frac{1}{a}\begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & \frac{-1}{2} & 0 & \frac{1}{2} \\ -1 & 0 & 1 & 0 \\ 0 & 0 & \frac{-1}{2} & \frac{1}{2} \\ 0 & 1 & -1 & 0 \\ \frac{-1}{4} & 0 & 0 & \frac{1}{4} \end{bmatrix} \begin{bmatrix} v \\ v - \frac{4}{11}as \\ v - \frac{4}{11}as \\ v - \frac{12}{11}as \end{bmatrix} = \begin{bmatrix} \frac{4}{11}s \\ \frac{4}{11}s \\ \frac{4}{11}s \\ \frac{4}{11}s \\ 0 \\ \frac{3}{11}s \end{bmatrix}$$