2018 Fall EECS205003 Linear Algebra - Midterm 1 sol.

Name: ID:

1. (a) $\|u\| = \sqrt{2}$, $\|v\| = \sqrt{2}$ and $\theta = 60°$

   (b) $\begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}$

   (c) $w = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$

   (d) Let $q(x) = a_0 + a_1 x + a_2 x^2 \in P_2$

   $< p, q >= 2a_0 + 3a_1 + 3a_2 = 0$, $a_0 = -1$, $a_1 = 0$ and $a_2 = 4$

   No, $< p, q > \neq 0$ so that $p(x)$ isn't orthogonal to $q(x)$.

2. (a) A $2 - D$ plane, or linear combination of the columns of A

   (b) All $\mathbf{b} \in \mathbf{C}(A)$.

   (c) The first column or the second column. (Or the third column if the condition is written clearly.)

3. (a) Yes

   (b) No, cause it doesn't include $\mathbf{0}$

   (c) Yes

   (d) No, cause $\mathbf{0}$ is not included

   (e) Yes

   (f) Yes

4. (a) Step1: Subtract three times the first row of A from the second row, and subtract the first row from the third.

   Step 2: Subtract twice the second row from the third.

$$\text{A} = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 6 \\ 1 & 4 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 7 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} = \text{U}$$

$$\therefore \text{L} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

   We solve $\text{A}\mathbf{x} = \begin{bmatrix} 5 \\ 8 \\ -12 \end{bmatrix}$ in two steps. First we solve $\text{L}\mathbf{y}=\mathbf{b}$ which in the case is

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 8 \\ -12 \end{bmatrix}$$

and we get $\mathbf{y_1 = 5}$; $15 + y_2 = 8$ so $\mathbf{y_2 = -7}$; and $5 - 14 + y_3 = -3$, so $\mathbf{y_3 = -3}$
Finally we solve U**x**=**y** which in the case is

$$\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -7 \\ -3 \end{bmatrix}$$

Here we have $\mathbf{x_3 = -3}$;$x_2 - 9 = -7$, so $\mathbf{x_2 = 2}$; $x_1 + 4 - 3 = 5$, so $\mathbf{x_1 = 4}$

(b) Here we have to reduce [L I] to [I L^{-1}] by row operations,

$$\begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 3 & 1 & 0 & & 0 & 1 & 0 \\ 1 & 2 & 1 & & 0 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & -3 & 1 & 0 \\ 0 & 2 & 1 & & -1 & 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & & 1 & 0 & 0 \\ 0 & 1 & 0 & & -3 & 1 & 0 \\ 0 & 0 & 1 & & 5 & -2 & 1 \end{bmatrix}$$

So $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}$

$\therefore U^{-1} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ by row operations [U I] to [I U^{-1}].

Since A=LU and L and U are both invertible, we have $A^{-1} = U^{-1}L^{-1}$.

$$A^{-1} = \begin{bmatrix} 1 & -2 & 5 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix} = \begin{bmatrix} 32 & -12 & 5 \\ -18 & 7 & -3 \\ 5 & -2 & 1 \end{bmatrix}$$

(c) To convert to a matrix form, use the general format A**x**= **b**:

$$\begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ 12 \end{bmatrix}$$

Subract three times the first row from the second row to get:

$$\text{A}=\begin{bmatrix} 2 & 3 \\ 6 & 15 \end{bmatrix} \rightarrow U = \begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix}.$$

Doing the same to the right side b =(5,12) gives a new equation of the form U**x**=**c**:

$$\begin{bmatrix} 2 & 3 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

To solve our new equation,

$$6\text{y} = \text{-3} \rightarrow \mathbf{y} = -\tfrac{\mathbf{1}}{\mathbf{2}}$$

$$2\text{x}+3\text{y}=5 \rightarrow 2\text{x}+3(-\tfrac{1}{2})=5 \rightarrow \mathbf{x} = \tfrac{\mathbf{13}}{\mathbf{4}}$$

(d) Let permutation matrix P $= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

For any permutation matrix PP^T,

$$P_{ij} = \begin{cases} 1, & \text{if } i = j. \\ 0, & \text{if } i \neq j. \end{cases} \tag{1}$$

Only in the diagonals the above condition is satisfied , so this proves that PP^T=I.

$$\mathrm{P}P^T = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}$$

(e) There are two 3*3 permutation matrices,

i) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ ii) $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$

5. (a) $l_{21} = 1$ $l_{31} = -2$ $l_{32} = 1$

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

(b) $U = \begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -1 \\ 0 & 0 & 6 \end{bmatrix}$ $\mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$

(c) $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 1 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} -11 \\ -13 \\ 8 \end{bmatrix}$ $\mathbf{c} = \begin{bmatrix} -11 \\ -2 \\ -12 \end{bmatrix}$

$$\begin{bmatrix} 1 & -1 & 4 \\ 0 & -1 & -1 \\ 0 & 0 & 6 \end{bmatrix} \mathbf{x} = \begin{bmatrix} -11 \\ -2 \\ -12 \end{bmatrix} \quad \mathbf{x} = \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$$

6. (a) $M^2 = \begin{bmatrix} A^2 + UV & AU + UD \\ VA + DV & VU + D^2 \end{bmatrix}$

(b) $WA = I_m$, $XA + I_nV = 0$, $WU = Y$, $XU + I_nD = Z$

$\because$ A is invertible

$\therefore$ $W = A^{-1}$, $X = -VA^{-1}$, $Y = A^{-1}U$, $Z = -VA^{-1}U + D$

(c) $\begin{bmatrix} I_m & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} M & P \\ N & Q \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$

$M = I_m$ and $N = 0$ (Can use simple matrix and Gauss-Jordan Elimination method to think)

$$\begin{bmatrix} I_m & Y \\ 0 & Z \end{bmatrix} \begin{bmatrix} I_m & P \\ 0 & Q \end{bmatrix} = \begin{bmatrix} I_m & 0 \\ 0 & I_n \end{bmatrix}$$

so we can get $I_mP + YQ = 0$, $ZQ = I_n$

because the Z is invertible, $Q = Z^{-1}$, $P = -YQ = -YZ^{-1}$

$$\begin{bmatrix} I_m & Y \\ 0 & Z \end{bmatrix}^{-1} = \begin{bmatrix} I_m & -YZ^{-1} \\ 0 & Z^{-1} \end{bmatrix}$$

use (b) can get :

$$\begin{bmatrix} I_m & -A^{-1}U(D - VA^{-1}U)^{-1} \\ 0 & (D - VA^{-1}U)^{-1} \end{bmatrix}$$

(d) use (b) can get :

$$\begin{bmatrix} A^{-1} & 0 \\ -VA^{-1} & I_n \end{bmatrix} \begin{bmatrix} A & U \\ V & D \end{bmatrix} = \begin{bmatrix} I_m & A^{-1}U \\ 0 & D - VA^{-1}U \end{bmatrix}$$

$$\begin{bmatrix} A & U \\ V & D \end{bmatrix}^{-1} = \begin{bmatrix} I_m & A^{-1}U \\ 0 & D - VA^{-1}U \end{bmatrix}^{-1} \begin{bmatrix} A^{-1} & 0 \\ -VA^{-1} & I_n \end{bmatrix}$$

use (c)

$$\begin{bmatrix} A & U \\ V & D \end{bmatrix}^{-1} = \begin{bmatrix} I_m & -A^{-1}U(D - VA^{-1}U)^{-1} \\ 0 & (D - VA^{-1}U)^{-1} \end{bmatrix} \begin{bmatrix} A^{-1} & 0 \\ -VA^{-1} & I_n \end{bmatrix}$$

$$= \begin{bmatrix} A^{-1} + A^{-1}U(D - VA^{-1}U)^{-1}VA^{-1} & -A^{-1}U(D - VA^{-1}U)^{-1} \\ -(D - VA^{-1}U)^{-1}VA^{-1} & (D - VA^{-1}U)^{-1} \end{bmatrix}$$

7. (a) To solve **S**, firstly we model **S**. The simplest one is to assume that S as a linear model, where $y = \theta_a x_a + \theta_b x_b + \theta_c x_c$ with real-number parameters. Write down the equations for the three experiments in matrix form.

By samples, we can list up three equations
$$\begin{aligned} \theta_a + 2\theta_b + 3\theta_c &= 1 \\ 2\theta_a + \theta_b + 2\theta_c &= 1 \\ 3\theta_a + 2\theta_b + \theta_c &= 1 \end{aligned}$$

The matrix form is $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(b) By elimination

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{\text{row2 - 2 row1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 3 & 2 & 1 \end{bmatrix} \xrightarrow{\text{row2 - 3 row1}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & -4 & -8 \end{bmatrix} \xrightarrow{\text{row3 - 4/3 row2}} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}$$

Then we have three elimination matrices

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{4}{3} & 1 \end{bmatrix},$$

Elimination matrix E is then

$$E = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{1}{3} & -\frac{4}{3} & 0 \end{bmatrix}$$

and decomposed matrix L and U are

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & \frac{4}{3} & 1 \end{bmatrix}, U = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix}$$

(c) Let $X = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{bmatrix}$, $\theta = \begin{bmatrix} \theta_a \\ \theta_b \\ \theta_c \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$EX\theta = E\mathbf{b} \Rightarrow U\theta = \begin{bmatrix} 1 \\ -1 \\ -\frac{2}{3} \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -4 \\ 0 & 0 & -\frac{8}{3} \end{bmatrix} \theta \Rightarrow \begin{aligned} \theta_c &= -\tfrac{3}{8}(-\tfrac{-2}{3}) = \tfrac{1}{4} \\ \theta_b &= \tfrac{-1+4\theta_c}{-3} = 0 \\ \theta_a &= 1 - 2\theta_b - 3\theta_c = \tfrac{1}{4} \end{aligned}$$

(d) Let matrix $X = \begin{bmatrix} -\mathbf{x}_1^T- \\ -\mathbf{x}_2^T- \\ -\mathbf{x}_3^T- \end{bmatrix}$ has rows representing sample input vector $\begin{bmatrix} x_a \\ x_b \\ x_c \end{bmatrix}$

If X has some rows that is linear dependent to others, but the corresponding outputs are not, then S has no solution. For example, change X to $\begin{bmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ but let y remain the same as on the problem. Since some rows are linear dependent, this indicates after elimination # of pivots < 3. Then inverse of X does not exist and system S is not solvable.

If y changes to $2, 1$ and 3 in this case, S is still solvable but we cannot find the unique parameters.