2018 Fall EECS205003 Linear Algebra - Midterm

1. (13%) Consider two vectors $\mathbf{u} = \begin{bmatrix}1\\0\\1\end{bmatrix} \in R^3$ and $\mathbf{v} = \begin{bmatrix}0\\1\\1\end{bmatrix} \in R^3$.

   (a) (3%) Find $\|\mathbf{u}\|$, $\|\mathbf{v}\|$ and the angle θ between $\mathbf{u}$ and $\mathbf{v}$.

   (b) (3%) Find the perpendicular projection of $\mathbf{v}$ on $\mathbf{u}$.
   (Hint: $\frac{<\mathbf{v},\mathbf{u}>}{\|\mathbf{u}\|^2}\mathbf{u}$)

   (c) (4%) Find a vector $\mathbf{w} \in R^3$ such that $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$.

   (d) (3%) Consider the space P_2 with inner product:
   $$< p, q >= p(0)q(0) + p(\tfrac{1}{2})q(\tfrac{1}{2}) + p(1)q(1).$$
   Is $p(x) = 4x^2 - 4x + 1$ orthogonal to $q(x) = 4x^2 - 1$? Please give your reason.
   (**Hint:** $P_n \equiv \{f(x)|f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,\ a_i \in R\}$)

2. (10%) For $A\mathbf{x} = \mathbf{b}$ where $A = \begin{bmatrix}1 & 4 & 6\\2 & 8 & 7\\-1 & -4 & 6\\3 & 12 & 7\end{bmatrix}$, and $\mathbf{b} = \begin{bmatrix}b_1\\b_2\\b_3\\b_4\end{bmatrix}$:

   (a) (3%) Describe the column space $\mathbf{C}(A)$ of A. What is the rank of A?

   (b) (4%) For which $\mathbf{b}$ (find a condition on b_1, b_2, b_3, b_4) is $A\mathbf{x} = \mathbf{b}$ solvable?

   (c) (3%) If we remove a column from A and $A\mathbf{x} = \mathbf{b}$ is still solvable, which column would it be? Please give your reason.

3. (12%) Test if the following sets of vectors form subspaces .

   (a) (2%) $\{\ (x_1, x_2, x_3) \in R^3\ :\ x_1 + 2x_2 + 3x_3 = 0\ \}$

   (b) (2%) $\{\ (x_1, x_2, x_3) \in R^3\ :\ x_1 + 2x_2 + 3x_3 = 4\ \}$

   (c) (2%) The set of all symmetric matrix.

   (d) (2%) The set of all nonsingular matrix.

   (e) (2%) $\{\ \begin{bmatrix}a & 0\\b & c\end{bmatrix}\ \in\ M\ |\ a + b + c = 0\ \}$

   (f) (2%) $\{\ \begin{bmatrix}a & 0\\b & c\end{bmatrix}\ \in\ M |\ \forall\ a,\ b,\ c \in R\ \}$

4. (15%) Let matrix $A = \begin{bmatrix}1 & 2 & 1\\3 & 7 & 6\\1 & 4 & 8\end{bmatrix}$.

   (a) (4%) Find the LU factorization of A. Use the answer to solve $\mathrm{A}\mathbf{x} = \begin{bmatrix}5\\8\\-12\end{bmatrix}$.

   (b) (4%) Use Gauss-Jordan Elimination method to find L^{-1} and U^{-1}.
   Use the answers to find A^{-1}.

Please turn over to continue.

(c) (3%) Consider the following linear equations, solve for x and y by using elimination and back substitution.

$$2x + 3y = 5$$
$$6x + 15y = 12$$

(d) (2%) Prove for any permutation matrix P, $PP^T = I$.

(e) (2%) Find 3×3 permutation matrices with $P^3 = I$ (but not $P = I$).

5. (18%) Consider the system of linear equations:

$$\begin{aligned} x - y + 4z &= -11 \\ x - 2y + 3z &= -13 \\ -2x + y - 3z &= 8 \end{aligned}$$

(a) (6%) For elimination, find l_{21}, l_{31}, l_{32} and E_{21}, E_{31}, E_{32}.

(b) (6%) Use $EA = U$ to get U and solve for $\mathbf{x}$.

(c) (6%) Use $A = LU$, solve $L\mathbf{c} = \mathbf{b}$ and $U\mathbf{x} = \mathbf{c}$ sequentially to get $\mathbf{c}$ and $\mathbf{x}$.

6. (12%) Consider matrix $M = \begin{bmatrix} A_{m\times m} & U_{m\times n} \\ V_{n\times m} & D_{n\times n} \end{bmatrix}$, please answer the following questions:

(a) (4%) Please use block matrices A, U, V, D to find M^2

(b) (4%) Suppose A is invertible and I_m, I_n are $m \times m$ and $n \times n$ identity matrices. Please use block matrices A, V, U, D or thier inverses to find $W_{m\times m}$, $X_{n\times m}$, $Y_{m\times n}$, and $Z_{n\times n}$ satisfying

$$\begin{bmatrix} W & 0 \\ X & I_n \end{bmatrix} \begin{bmatrix} A & U \\ V & D \end{bmatrix} = \begin{bmatrix} I_m & Y \\ 0 & Z \end{bmatrix}.$$

(c) (4%) Suppose Z is invertible. Find the inverse of $\begin{bmatrix} I_m & Y \\ 0 & Z \end{bmatrix}$.

(d) (4%) Use the results of (a) and (b) to compute the inverse of $\begin{bmatrix} A & U \\ V & D \end{bmatrix}$.

7. (20%) Given a system $\mathbf{S}: y = f(x_a, x_b, x_c)$, where $x_a, x_b, x_c \in \mathbb{R}$ are inputs and $y \in \mathbb{R}$ is the output. By experiments, we get three samples whose inputs are $(x_a, x_b, x_c) = (1, 2, 3)$, $(2, 1, 2)$ and $(3, 2, 1)$, and output $y = 1, 1$ and 1 respectively. Try to answer following questions:

(a) (2%) To solve $\mathbf{S}$, firstly we model $\mathbf{S}$. The simplest one is to assume that S as a linear model, where $y = \theta_a x_a + \theta_b x_b + \theta_c x_c$ with real-number parameters. Write down the equations for the three experiments in matrix form.

(b) (12%) According to (a), what are the elimination matrix E and decomposed matrices L and U?

(c) (2%) Please solve θ_a, θ_b and θ_c.

(d) (4%) Under what condition $\mathbf{S}$ does have no solution? Why? If we change y to $2, 1$ and 3, is there solution to $\mathbf{S}$ under that condition?