2018 Fall EECS205003 Linear Algebra - Homework 5 Solution

Name: ID:

1. $A = S\Lambda S^{-1}$
   Diagonalize A,use $S\Lambda^k S^{-1}$ to find A^k

   (a) $A_1 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$

   (b) $A_2 = \begin{bmatrix} 5 & -4 & 4 \\ 12 & -11 & 12 \\ 4 & -4 & 5 \end{bmatrix}$

   Solution:

   (a) $\lambda = 1, 3$ and $\Lambda = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}$, $S = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, $S^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $A_1^k = \frac{1}{2}\begin{bmatrix} 1+3^k & 1-3^k \\ 1-3^k & 1+3^k \end{bmatrix}$

   (b) $\lambda = 1, 1, -3$ and $\Lambda = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$, $S = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{bmatrix}$, $S^{-1} = \begin{bmatrix} 3 & -2 & 3 \\ 1 & -1 & 2 \\ -1 & 1 & -1 \end{bmatrix}$,
   $A_2^k = \begin{bmatrix} 2-(-3)^k & -1+(-3)^k & 1-(-3)^k \\ 3-3*(-3)^k & -2+3*(-3)^k & 3+(-3)^{k+1} \\ 1-(-3)^k & (-1)+(-3)^k & 2-(-3)^k \end{bmatrix}$

2. Assume there are 3 slotting machine A and B. The chance to win the reward at A and B is 40% and 30%, respectively. If you win, you stay at the same machine. But if you lose, you choose other machines. Please answer the following questing.

   (a) Try to model relation between the chance to choose A or B at $t+1$ playing times and at t times where $t \geq 0$.

   (b) Suppose you randomly choose them at the same chance at first. After playing for a long time, what is the chance to choose A or B? (hint: $t \to \infty$)

   Solution:

   (a) $\det\begin{bmatrix} 10-\lambda & 3i \\ -3i & 2-\lambda \end{bmatrix} = \lambda^2 - 12\lambda + 20 - 9 = \lambda^2 - 12\lambda + 11 = 0 \to \lambda = 1, 1, 1$

   $\lambda = 1 \to x_1 = c_1 \begin{bmatrix} i \\ -3 \end{bmatrix} \to E_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} i \\ -3 \end{bmatrix}$

   $\lambda = 11 \to x_2 = c_2 \begin{bmatrix} 3i \\ 1 \end{bmatrix} \to E_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 3i \\ 1 \end{bmatrix}$

   (b) $U = \begin{bmatrix} E_1 & E_2 \end{bmatrix} = \begin{bmatrix} \frac{i}{\sqrt{10}} & \frac{3i}{\sqrt{10}} \\ \frac{-3}{\sqrt{10}} & \frac{1}{\sqrt{10}} \end{bmatrix} \to AU = U\Lambda \to U^{+}AU = \Lambda = \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix}$

3. A and B have the same eigenvalues, which are $\lambda_1 = -2, \lambda_2 = -2, \lambda_3 = 4$.

(a) Find each matrix eigenvectors.

(b) Which matrix can be diagonalized, and why?

$$A = \begin{bmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{bmatrix} \quad B = \begin{bmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{bmatrix}$$

Solution:

(a) For A matrix

$$\lambda = -2 \rightarrow \begin{bmatrix} 3 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \rightarrow x_1 - x_2 + x_3 = 0$$

$$\rightarrow \mathbf{x_1} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = 4 \rightarrow \begin{bmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \mathbf{x_2} = c_3 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \rightarrow \text{eigenvectors:} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$$

For B matrix

$$\lambda = -2 \rightarrow \begin{bmatrix} -1 & 1 & -1 \\ -7 & 7 & -1 \\ 6 & -6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \mathbf{x_1} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\lambda = 4 \rightarrow \begin{bmatrix} -7 & 1 & -1 \\ -7 & 1 & -1 \\ -6 & 6 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\rightarrow \mathbf{x_3} = c_2 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \text{eigenvectors:} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(b) Although each matrix has repeat eigenvalues, it may or may not have independent eigenvectors. A matrix has 3 independent eigenvectors$\rightarrow$ A is diagonalizable. On the other hand, B matrix only has 2 independent eigenvectors$\rightarrow$ B is not diagonalizable.

4. (a) Find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} 10 & 3i \\ -3i & 2 \end{bmatrix}$$

(b) Construct a 2×2 matrix U such that $U^{+}AU = \Lambda$, where $U^{+} = (U*)^{T}$ is the complex-conjugate transpose of U and Λ is a real diagonal matrix. (Hint: relate U to the unit-length eigenvectors of A and relate Λ to the eigenvalues of A.)

5. Find all eigenvalues of each A

(a) $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$

(b) $A = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 1 & 1 & 2 & 2 \\ 2 & 2 & 1 & 1 \\ 2 & 2 & 1 & 1 \end{bmatrix}$

Solution:

(a) $\lambda = 1, 2, 3$

(b) $\lambda = 0, 0, 6, -2$

6. Solve the following differential equation systems:

$y_1' = -3y_1 - y_2 + 2y_3$
$y_2' = -4y_2 + 2y_3$
$y_3' = y_2 - 5y_3$

Solution:

$$\mathbf{y} = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} e^{-3t} + c_3 \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} e^{-6t}$$