2017 Fall EE203001 Linear Algebra - Homework 5 solution

Due: 2017/12/8

1. (14%) Find two λ and $\boldsymbol{x}$ so that $\boldsymbol{y} = e^{\lambda t}\boldsymbol{x}$ solve

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}.$$

What combination $\boldsymbol{y} = c_1 e^{\lambda_1 t}\boldsymbol{x}_1 + c_2 e^{\lambda_2 t}\boldsymbol{x}_2$ starts from $\boldsymbol{y}(0) = (0,1) =?$

Solution:

$$\det \begin{bmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{bmatrix} = 0$$

$$\rightarrow \lambda = -3, \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \boldsymbol{x} = \mathbf{0},\ \boldsymbol{x}_1 = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\lambda = -1, \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \boldsymbol{x} = \mathbf{0},\ \boldsymbol{x}_2 = c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y(t) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t} \rightarrow y(0) = c_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\rightarrow \begin{cases} c_1 + c_2 = 0 \\ -c_1 + c_2 = 1 \end{cases} \rightarrow \begin{cases} c_1 = \frac{-1}{2} \\ c_2 = \frac{1}{2} \end{cases}$$

$$\rightarrow y(t) = \frac{-1}{2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-3t} + \frac{1}{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{-t}$$

2. (14%) Find A to change the scalar equation $y'' = 4y' - 3y$ into a vector equation for $\boldsymbol{u} = (y, y')$:

$$\frac{d\boldsymbol{y}}{dt} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix} = A\boldsymbol{u}$$

What are the eigenvalues of A? Find them also by substituting $y = e^{\lambda t}$ into $y'' = 4y' - 3y$.

Solution:

$$\frac{d}{dt} \begin{bmatrix} y \\ y' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix} \begin{bmatrix} y \\ y' \end{bmatrix}$$

$(i)\ A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix},\ det(A - \lambda I) = 0 \rightarrow \lambda^2 - 4\lambda + 3 = 0,\ \lambda = 3, 1$

$(ii)\ \lambda^2 e^{\lambda t} = 4\lambda e^{\lambda t} - 3e^{\lambda t},\ \rightarrow \lambda^2 = 4\lambda - 3,\ \lambda = 3, 1$

3. (12%) Diagonalize the Matrix $B = \begin{bmatrix} n & 1 \\ 0 & n-1 \end{bmatrix}$. If $n = 2$, find b_{12} of B^{10}.

Solution:

$$B = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} n & 0 \\ 0 & n-1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$$

$$\to B^k = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} n^k & 0 \\ 0 & (n-1)^k \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} n^k & n^k - (n-1)^k \\ 0 & (n-1)^k \end{bmatrix}$$

Given $n = 2$, b_{12} of $B^{10} = 2^{10} - 1 = 1023$

4. (10%) Suppose G_{k+2} is the average of the two previous numbers G_{k+1} and G_k:

$$\begin{aligned} G_{k+2} &= \tfrac{1}{2}G_{k+1} + \tfrac{1}{2}G_k \\ G_{k+1} &= G_k \end{aligned} \qquad \text{is} \qquad \begin{bmatrix} G_{k+2} \\ G_{k+1} \end{bmatrix} = \begin{bmatrix} A \end{bmatrix} \begin{bmatrix} G_{k+1} \\ G_k \end{bmatrix}$$

(a) Find the eigenvalues and eigenvectors of A.

(b) Find the limit as $n \to \infty$ of the matrices $A^n = S\Lambda^n S^{-1}$.

Solution:

(a) $A = \begin{bmatrix} 0.5 & 0.5 \\ 0 & 1 \end{bmatrix}$, has $\lambda_1 = 1$, $\lambda_2 = 0.5$ with $x_1 = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $x_2 = c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

(b) $A^n = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1^n & 0 \\ 0 & (0.5)^n \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \to A^\infty = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$

5. (14%) Solve these linear equation by Cramer's Rule $\boldsymbol{x_j} = det(\boldsymbol{B_j}) \det(\boldsymbol{A})$

(a) $\begin{aligned} x_1 + 2x_2 &= 3 \\ 4x_1 + 5x_2 &= 6 \end{aligned}$ (b) $\begin{aligned} x_1 + 2x_2 \qquad &= 0 \\ 2x_1 + x_2 + 3x_3 &= 1 \\ 3x_2 + x_3 &= 0 \end{aligned}$

Solution:

(a) $\begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} = -3$, $\begin{bmatrix} 3 & 2 \\ 6 & 5 \end{bmatrix} = 3$, $\begin{bmatrix} 1 & 3 \\ 4 & 6 \end{bmatrix} = -6$

$x_1 = -\frac{3}{3} = -1$, $x_2 = \frac{-6}{-3} = 2$

(b) $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 3 \\ 0 & 3 & 1 \end{vmatrix} = -12$, $\begin{vmatrix} 0 & 2 & 0 \\ 1 & 1 & 3 \\ 0 & 3 & 1 \end{vmatrix} = -2$, $\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & 3 \\ 0 & 0 & 1 \end{vmatrix} = 1$, $\begin{vmatrix} 1 & 2 & 0 \\ 2 & 1 & 1 \\ 0 & 3 & 0 \end{vmatrix} = -3$

$x_1 = -\frac{-2}{-12} = \frac{1}{6}$, $x_2 = \frac{1}{-12}$, $x_3 = \frac{-3}{-12} = \frac{1}{4}$

6. (12%) From the formula $\boldsymbol{AC^T} = (det\boldsymbol{A})\boldsymbol{I}$ show $(det\boldsymbol{A})^{n-1}$

Solution:

$$det(c)$$

$$= det(det(A)(A^{-1})^T)$$

$$= [det(A)]^n det((A^{-1})^T)$$

$$= [det(A)]^n det(A^{-1})$$

$$= [det(A)]^n \frac{1}{det(A)}$$

$$= [det(A)]^{n-1}$$

7. (12%) The Block $\boldsymbol{B}$ has eiqencalue 3.5 and $\boldsymbol{C}$ has eiqencalue 1.2, and $\boldsymbol{D}$ has eigenvalues 3.9

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{B} & \boldsymbol{C} \\ \boldsymbol{O} & \boldsymbol{D} \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 & 1 \\ 1 & 5 & -2 & 3 \\ 0 & 0 & 6 & 3 \\ 0 & 0 & 3 & 6 \end{bmatrix}, \text{ Find the eigenvalue of } \boldsymbol{A - 2I}.$$

Solution:

The block matrix has $\lambda = 3, 5$ form B and $3, 9$ from D, so the eigenvalues of A are $3, 5, 3, 9$.

$\rightarrow$The eigenvalues of $A - 2I$ are $1, 3, 1, 7$.

8. (12%) $\boldsymbol{A} = \begin{bmatrix} 3 & -2 & 2 \\ 6 & -4 & 6 \\ 2 & -1 & 3 \end{bmatrix}$

(a) Find the eigenvalues and eigenvector of $\boldsymbol{A}$.
(b) Diagonalize the matrix $\boldsymbol{A}$.

Solution:

$$\text{(a)} \ det(A - \lambda I) = \begin{bmatrix} 3-\lambda & -2 & 2 \\ 6 & -4-\lambda & 6 \\ 2 & -1 & 3-\lambda \end{bmatrix} \rightarrow \lambda = 1, -1, 2$$

$$\lambda = 1 \rightarrow \begin{bmatrix} 2 & -2 & 2 \\ 6 & -5 & 6 \\ 2 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ x_1 = c_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = -1 \rightarrow \begin{bmatrix} 4 & -2 & 2 \\ 6 & -3 & 6 \\ 2 & -1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ x_2 = c_2 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

$$\lambda = 2 \rightarrow \begin{bmatrix} 1 & -2 & 2 \\ 6 & -6 & 6 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ x_3 = c_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

(b) $AP = \begin{bmatrix} A\boldsymbol{x_1} & A\boldsymbol{x_2} & A\boldsymbol{x_3} \end{bmatrix}$

$$= \begin{bmatrix} \boldsymbol{x_1} & -\boldsymbol{x_2} & 2\boldsymbol{x_3} \end{bmatrix}$$

$$= \begin{bmatrix} \boldsymbol{x_1} & \boldsymbol{x_2} & \boldsymbol{x_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} = PD$$

$A = PDP^{-1}$

$$= \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -1 & 1 & 0 \\ 0 & 2 & 1 \\ 1 & 0 & 1 \end{bmatrix}^{-1}$$