2017 Fall EE203001 Linear Algebra - Homework 4 solution

Due: 2017/11/24

1. (10%) For the closest parabola $b = C + Dt + Et^2$ to the four points $(0, 0)$, $(1, 8)$, $(3, 8)$, and $(4, 20)$. Write down the unsolvable equations $A\mathbf{x} = \mathbf{b}$ in three unknowns $\mathbf{x} = (C, D, E)$. Set up the three normal equations $A^T A\mathbf{x} = A^T\mathbf{b}$ (solution not required).

   Solution:

   Parabola Project b 4D to 3D:

$$A\hat{\mathbf{x}} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 0 \\ 8 \\ 8 \\ 20 \end{bmatrix}.$$

$$A^T A\hat{\mathbf{x}} = \begin{bmatrix} 4 & 8 & 26 \\ 8 & 26 & 92 \\ 26 & 92 & 338 \end{bmatrix} \begin{bmatrix} C \\ D \\ E \end{bmatrix} = \begin{bmatrix} 36 \\ 112 \\ 400 \end{bmatrix}.$$

2. (10%) Use row operations to simplify and compute these determinants:

$$\det \begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix} \text{ and } \det \begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix}.$$

   Solution:

$$\begin{bmatrix} 101 & 201 & 301 \\ 102 & 202 & 302 \\ 103 & 203 & 303 \end{bmatrix} \rightarrow \begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 2 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 101 & 201 & 301 \\ 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & t & t^2 \\ t & 1 & t \\ t^2 & t & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & t-t^3 & 1-t^4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & t & t^2 \\ 0 & 1-t^2 & t-t^3 \\ 0 & 0 & 1-t^2 \end{bmatrix}$$

   The first determinant is 0 and the second is $1 - 2t^2 + t^4 = \left(1 - t^2\right)^2$.

3. (10%) Given $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 1 & 2 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix}$

   (a) Find orthonormal vectors $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{q}_3$ such that $\mathbf{q}_1$, $\mathbf{q}_2$ span the column space of $\mathbf{A}$.

   (b) Which of the four fundamental subspace contains $\mathbf{q}_3$?

   (c) Solve $\mathbf{Ax} = \mathbf{b}$ by least squares.

   Solution:

   (a) $\mathbf{v}_1 = \mathbf{a}_1 \rightarrow \|\mathbf{v}_1\|^2 = 2$

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a_2^T v_1}}{\|\mathbf{v_1}\|^2} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} - \frac{2}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \rightarrow \|\mathbf{v}_2\|^2 = 2$$

$$A^T = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\mathbf{v}_3 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \rightarrow \|\mathbf{v}_3\|^2 = 3$$

$$\mathbf{q}_1 = \frac{\mathbf{v_1}}{\|\mathbf{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \mathbf{q}_2 = \frac{\mathbf{v_2}}{\|\mathbf{v}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \quad \mathbf{q}_3 = \frac{\mathbf{v_3}}{\|\mathbf{v}_3\|} = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

(b) The null space of A^T contains $\mathbf{q_3}$

(c) $\hat{x} = (A^TA)^{-1}A^Tb = (4, -2)$

4. (10%) Find the determinant of A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 4 & 3 & 5 \\ 4 & 16 & 9 & 25 \\ 8 & 64 & 27 & 125 \end{bmatrix}$$

Solution:

A is 4 by 4 ”Vandermonde’s matrix”.

$det(A) = (4-2)(3-2)(5-2)(3-4)(5-4)(5-3) = 2 \cdot 1 \cdot 3 \cdot (-1) \cdot 1 \cdot 2 = -12$

5. (10%) Use the big formula with n! terms: $|A| = \sum \pm a_{1\alpha}a_{2\beta}....a_{n\omega}$ to compute the determinants of A, B, C from six terms. Are their rows independent?

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 5 & 1 & 3 \\ 5 & 3 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & 2 \\ 3 & 4 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Solution:

$det(A) = 1 + 45 + 75 - 9 - 15 - 25 = 72$ rows are independent

$det(B) = 10 + 12 + 24 - 8 - 20 - 18 = 0$ row1+row2=row3 rows are dependent

$det(C) = -1$ rows are independent

6. (10%) Use cofactors $C_{ij} = (-1)^{i+j} detM_{ij}$. Find all cofactors of A and B and put them into cofactor matrices C, D. What is $\det(B)$?

$$A = \begin{bmatrix} a & c \\ b & d \end{bmatrix}, B = \begin{bmatrix} 1 & 3 & 5 \\ 5 & 1 & 3 \\ 1 & 0 & 0 \end{bmatrix}$$

Solution:

$$C = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, D = \begin{bmatrix} 0 & 15 & -5 \\ 0 & -25 & 15 \\ 4 & 22 & -14 \end{bmatrix}, det(B) = 4$$

7. (10%) Consider the matrix

$$A = \begin{bmatrix} 3 & 3 & b \\ -2 & a & 7 \\ 9 & 5 & c \end{bmatrix} = LU \quad \text{where } U = \begin{bmatrix} 1 & d & e \\ 0 & -2 & f \\ 0 & 0 & 3 \end{bmatrix}$$

Please find the determinants of $L, U, A, U^{-1}L^{-1}$ and $U^{-1}L^{-1}A$.

Solution:

Note that the diagonal elements of L are 1's, so $\det(L) = 1 * 1 * 1 = 1$.

$\det(U) = 1 * (-2) * 3 = -6$, $\det(\text{A}) = \det(L)\det(U) = -6$.

$\det(U^{-1}L^{-1}) = \det(U^{-1})\det(L^{-1}) = 1 * (\frac{-1}{6}) = \frac{-1}{6}$. $\det(U^{-1}L^{-1}A) = \det(I) = 1$.

8. (10%) Compute the determinants of these matrix by proper row/column operation:

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix}$$

Solution:

For matrix A:

$$A = \begin{bmatrix} 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ d & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \\ 0 & a & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & 0 & c \\ 0 & 0 & b & 0 \end{bmatrix} \rightarrow \begin{bmatrix} d & 0 & 0 & 0 \\ 0 & a & 0 & 0 \\ 0 & 0 & b & 0 \\ 0 & 0 & 0 & c \end{bmatrix} = A'$$

Since we do three row exchanges to A, $\det(A') = abcd = (-1)^3\det(A) \Rightarrow \det(A) = -abcd$.
For matrix B:

$$B = \begin{bmatrix} a & a & a \\ a & b & b \\ a & b & c \end{bmatrix} \rightarrow \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & b-a & c-a \end{bmatrix} \rightarrow \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-a-(b-a) \end{bmatrix} \rightarrow \begin{bmatrix} a & a & a \\ 0 & b-a & b-a \\ 0 & 0 & c-b \end{bmatrix} = B'$$

Since subtracting a multiple of one row from another row doesn't change the determinant, $\det(B')$ $= \det(B) = a(b-a)(c-b)$.

9. (10%) Find $\boldsymbol{q}_1$, $\boldsymbol{q}_2$, $\boldsymbol{q}_3$ (orthonormal) as combinations of $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ (independent columns of A). Then write A as QR:

$$A = \begin{bmatrix} 0 & 3 & 6 \\ 1 & 2 & 4 \\ 0 & 0 & 2 \end{bmatrix}$$

Solution:

$$\mathbf{v}_1 = \mathbf{a}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \rightarrow \|\mathbf{v}_1\|^2 = 1$$

$$\mathbf{v}_2 = \mathbf{a}_2 - \frac{\mathbf{a_2^T v_1}}{\|\mathbf{v}_1\|^2} = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix} - \frac{2}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \rightarrow \|\mathbf{v}_2\|^2 = 9$$

$$\mathbf{v}_3 = \mathbf{a}_3 - \frac{\mathbf{a_3^T v_1}}{\|\mathbf{v}_1\|^2} - \frac{\mathbf{a_3^T v_2}}{\|\mathbf{v}_2\|^2} = \begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix} - \frac{4}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} - \frac{18}{9} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} \rightarrow \|\mathbf{v}_2\|^2 = 4$$

$$\mathbf{q}_1 = \frac{\mathbf{v_1}}{\|\mathbf{v}_1\|} = \frac{1}{1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{q}_2 = \frac{\mathbf{v_2}}{\|\mathbf{v}_2\|} = \frac{1}{3} \begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{q}_3 = \frac{\mathbf{v_3}}{\|\mathbf{v}_3\|} = \frac{1}{2} \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix}$$

$$A = QR = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 4 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

10. (10%) (a) Find a basis for the subspace $\mathbf{S}$ in $\mathbf{R}_4$ spanned by all solutions of $x_1 - x_2 - x_3 - x_4 = 0$ (b) Find a basis for the orthogonal complement $\mathbf{S}^\perp$. (c) Find $\boldsymbol{b}_1$ in $\boldsymbol{S}$ and $\mathbf{b}_2$ in $\boldsymbol{S}^\perp$ so that $\boldsymbol{b}_1 + \boldsymbol{b}_2 = h = (1, -1, -1, 1)$.

Solution:

(a) let $x_2 = c_1$, $x_3 = c_2$, $x_4 = c_3$,

$$\rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$\rightarrow \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ is the basis for the subspace $\boldsymbol{S}$.

(b) Since $\mathbf{S}$ contains solutions to $\begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \mathbf{x} = 0$, a basis for $\mathbf{S}^\perp = \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix}$

(c) Split (1, -1, -1, 1) = $\mathbf{b}_1 + \mathbf{b}_2$ by projection on $\mathbf{S}^\perp$ and $\mathbf{S}$: $\mathbf{b}_1 = (\frac{1}{2}, \frac{1}{-2}, \frac{1}{-2}, \frac{3}{2})$ and $\mathbf{b}_2 = (\frac{1}{2}, \frac{1}{-2}, \frac{1}{-2}, \frac{1}{-2})$