2017 Fall EE203001 Linear Algebra - Homework 3 solution

Due: 2017/11/10

1. (10%) Suppose $\mathbf{S}$ is spanned by the vectors (1, 2, 1, 4) and (1, 3, 3, 4). Find two vectors that span $\mathbf{S}^{\perp}$. This is the same as solving $A\mathbf{x} = \mathbf{0}$ for which A ?

   Solution: $\begin{bmatrix} 1 & 2 & 1 & 4 \\ 1 & 3 & 3 & 4 \end{bmatrix} \mathbf{x} = \mathbf{0} \Rightarrow \begin{bmatrix} 1 & 0 & -3 & 4 \\ 0 & 1 & 2 & 0 \end{bmatrix} \mathbf{x} = \mathbf{0}$

   Let $x_3 = c_1$ and $x_4 = c_2$. Then $x_1 = 3c_1 - 4c_2$ and $x_2 = -2c_1$.

   $S^{\perp}$=c_1(3, -2, 1, 0)+c_2(-4, 0, 0, 1)

   $\boldsymbol{S}^{\perp}$ is spanned by (3, -2, 1, 0) and (-4, 0, 0, 1).

2. (10%) Project the vector $\mathbf{b}$ onto $\mathbf{a}$ to find $\mathbf{p}$. Let $\mathbf{e} = \mathbf{b} - \mathbf{p}$ and show that $\mathbf{e}$ is perpendicular to $\mathbf{a}$.

   (a) $\mathbf{b}$=(1, 2, 3), $\mathbf{a}$=(1, 0, 1).

   (b) $\mathbf{b}$=(1, 3, 5, 7), $\mathbf{a}$=(0, 1, 0, 1).

   Solution:

   (a) $\hat{\boldsymbol{x}} = \frac{\mathbf{a^T b}}{\mathbf{a^T a}}$, $\boldsymbol{p} = \hat{\mathbf{x}}\mathbf{a}$

   $\hat{\boldsymbol{x}} = \frac{1+3}{1+1} = 2$, $\mathbf{p} = 2\mathbf{a}$, $\boldsymbol{e} = \boldsymbol{b} - \boldsymbol{p}$=(1, 2, 3)-(2, 0, 2)=(-1, 2, 1)

   $\boldsymbol{e} \cdot \boldsymbol{a} = -1 \times 1 + 2 \times 0 + 1 \times 1 = 0$

   (b) $\boldsymbol{p} = \mathbf{5}\boldsymbol{a}$, **e=b-p**=(1, 3, 5, 7)-(0, 5, 0, 5)=(0, -2, 5, 2)

   $\boldsymbol{e} \cdot \mathbf{a} = 0 \times 0 - 2 \times 1 + 5 \times 0 + 2 \times 1 = 0$

3. (12%) Project $\boldsymbol{b}$ = (0, 2, 8, 20) onto the line $\boldsymbol{a}$ = (2, 1, 1, 2). Find

   (a) $\hat{\boldsymbol{x}}$ = ?

   (b) Projection $\boldsymbol{p}$ = ?

   (c) Is $\boldsymbol{e}$ = $\boldsymbol{b}$ - $\boldsymbol{p}$ perpendicular $\boldsymbol{a}$? Please explain it.

   (d) $||\boldsymbol{e}||$

   Solution:

   (a) $\hat{\boldsymbol{x}} = \boldsymbol{a}^T\boldsymbol{b}/\boldsymbol{a}^T\boldsymbol{a} = 5$

   (b) Projection $\boldsymbol{p} = \hat{\boldsymbol{x}}\boldsymbol{a}$ = (10, 5, 5, 10)

   (c) Yes, $\boldsymbol{e}$ = $\boldsymbol{b}$-$\boldsymbol{p}$ = (-10, -3, 3, 10) , $\boldsymbol{e}^T\boldsymbol{a} = 0$

   (d) $||\boldsymbol{e}|| = \sqrt{218}$

4. (8%) Let $b = C + Dt$ be closest line to the points $(b, t) = (1, 2), (13, 4)$, and $(11, 3)$. Find the least squares solution $\hat{\boldsymbol{x}} = (C, D)$.

Solution:

$$Ax = \begin{bmatrix} 1 & -2 \\ 1 & 4 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 1 \\ 13 \\ 11 \end{bmatrix} = b.$$

The solution $\hat{\boldsymbol{x}} = \begin{bmatrix} 5 \\ 2 \end{bmatrix}$ comes from $A^T A\hat{\boldsymbol{x}} = \begin{bmatrix} 3 & 5 \\ 5 & 29 \end{bmatrix} \begin{bmatrix} C \\ D \end{bmatrix} = \begin{bmatrix} 25 \\ 83 \end{bmatrix} = A^T b$

5. (10%) Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$

(a) Find a basis for the null space of A.

(b) Given $\boldsymbol{x} = \begin{bmatrix} 2 & 3 & -1 \end{bmatrix}^T$, split it into a row space component $\boldsymbol{x_r}$ and $\boldsymbol{x_n}$.

Solution:

(a) $A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \to \begin{cases} x + z = 0 \\ y + z = 0 \end{cases}$

Let $z = c \to \begin{cases} x = -c \\ y = -c \end{cases} \quad \to \quad N(A) = \left\{ c \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$

$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \right\}$ is a basis for $N(A)$

(b) $\boldsymbol{x} = \boldsymbol{x_n} + \boldsymbol{x_r} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}$

Let $\boldsymbol{v} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, the projection of $\boldsymbol{x}$ onto the $\boldsymbol{v}$ (Null space of A) is $\boldsymbol{x_n}$

$$\boldsymbol{x_n} = \frac{\boldsymbol{x}^T \boldsymbol{v}}{\|\boldsymbol{v}\|^2} \boldsymbol{v} = \frac{6}{3} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

$$\to \boldsymbol{x_r} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

6. (12%) If A is a matrix and $W^\perp$ is the orthogonal complement of a vector set W, which of the following are false? Why?

   (a) $W^\perp$ is always a subspace.

   (b) $C(A)^\perp = C(A^T)$

   (c) $C(A) = N(A)^\perp$

   (d) $C(A)^\perp = N(A^T)$

   Solution:

   (a):

   Let $\mathbf{v}_1$,$\mathbf{v}_2 \in \mathbf{W}^\perp$,$\mathbf{v}_1 \perp \mathbf{W}$ and $\mathbf{v}_2 \perp \mathbf{W}$.

   For any $\mathbf{w} \in \mathbf{W}$,

   $\vec{0}^T w = 0 \to \vec{0} \in W^\perp$

   For any $\mathbf{w} \in \mathbf{W}$,

   $(\mathbf{v}_1 + \mathbf{v}_2)^T \mathbf{w} = \mathbf{v}_1^T \mathbf{w} + \mathbf{v}_2^T \mathbf{w}$ =0+0

   For any scalar c,

   $(c\mathbf{v})^T \mathbf{w} = c(\mathbf{v}^T \mathbf{w}) = c \cdot 0$

   So $\mathbf{W}^\perp$ is a subspace.

   (b), (c):

   $C(A)^\perp = N(A^T)$

   $C(A) = N(A^T)^\perp$

   $C(A^T)^\perp = N(A)$

   $C(A^T) = N(A)^\perp$

7. (8%) What linear combination of (-1, 1, 1) and (1, -1, 2) is closest to b = (3, -1, 7) ?

   Solution:

   $$A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 1 & 2 \end{bmatrix},\ A^T A\hat{x} = A^T b$$

   $$\to \hat{x} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \to b = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = A\hat{x} + e$$

8. (10%) Suppose A is the 4 by 4 identity matrix with its last column removed. A is 4 by 3. Project $\mathbf{b} = (1, 3, 4, 2)$ onto the column space of A. What shape is the projection matrix P and what is P ?

Solution:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \; P = \text{square matrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{p} = P\mathbf{b} = \begin{bmatrix} 1 \\ 3 \\ 4 \\ 0 \end{bmatrix}$$

9. (10%) Consider the matrix

$$P = \frac{1}{6} \begin{bmatrix} 5 & 2 & -1 \\ 2 & 2 & 2 \\ -1 & 2 & 5 \end{bmatrix}$$

Show that the length squared of column 2 equals P_{22}. Prove that the relation is true for any column n of P and P_{nn}. (Hint: use the properties of projection matrices).

Solution:

The length squared of column 2 is $\frac{4}{36} + \frac{4}{36} + \frac{4}{36} = \frac{2}{6} = P_{22}$.

Using the properties $P^T = P$ and $P^2 = P$, We have $P^T P = P$ so that for $P:$ (column n)^T(column n) = P_{nn}.

10. (10%) Please prove the statement: If $A^T A\mathbf{x} = 0$, then $A\mathbf{x} = 0$ by examine which subspaces $A\mathbf{x}$ shall fall into.

Solution:

If $A^T A\mathbf{x} = 0, A\mathbf{x}$ shall fall into the left nullspace of A. And clearly $A\mathbf{x}$ falls into the column space of A. The conclusion can be made that $A\mathbf{x}$ falls into both subspaces being perpendicular to each other only if $A\mathbf{x} = 0$.