EE205003 Session 9

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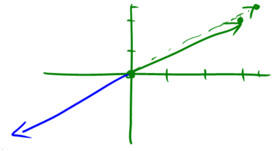
Def A vector space is a collection of vectors which is closed under lin. combinations

(For any **u** & **w** in the space, c**u**+d**w** is also in the space, c & d are any real numbers)

Ex :

R^2 : All 2D real vectors

$([3 \quad 2], [0 \quad 0], [\pi \quad e])$



More generally,

Def The space R^n consists of all col. vectors **v** with n real components (for complex components, we have C^n)

ADD & MUL need to follow 8 rules:

(1) $\mathbf{x}+\mathbf{y}=\mathbf{y}+\mathbf{x}$
(2) $\mathbf{x}+(\mathbf{y}+\mathbf{z}) = (\mathbf{x}+\mathbf{y})+\mathbf{z}$
(3) $\exists$ a unique zero vector $\mathbf{0}$ s.t. $\mathbf{x}+\mathbf{0} = \mathbf{x}$, $\forall$ $\mathbf{x}$
(4) For each $\mathbf{x}$, $\exists$ unique $-\mathbf{x}$ s.t. $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$
(5) $1\mathbf{x} = \mathbf{x}$
(6) $(c_1 c_2)\mathbf{x} = c_1(c_2\mathbf{x})$
(7) $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
(8) $(c_1 + c_2)\mathbf{x} = c_1\mathbf{x} + c_2\mathbf{x}$

Ex: if we define $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + y_2 \\ x_2 + y_1 \end{bmatrix}$

with usual mul. $c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \end{bmatrix}$

Do we still satisfy the 8 rules ?

No! $\mathbf{x} + \mathbf{y} \neq \mathbf{y} + \mathbf{x}$

$\mathbf{x} + (\mathbf{y} + \mathbf{z}) \neq (\mathbf{x} + \mathbf{y}) + \mathbf{z}$

$(c_1 + c_2)\mathbf{x} \neq c_1\mathbf{x} + c_2\mathbf{x}$

Other examples of vector spaces:

M : vector space of all real 2x2 matrices

F : vector space of all real functions f(x)

Z : vector space that consists only of a zero vector

We can ADD, MUL, still in vector space

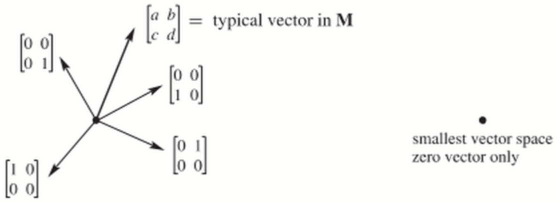
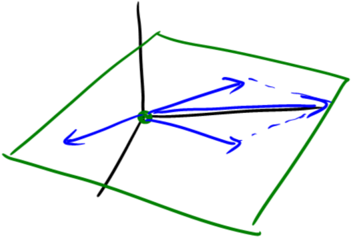


Figure 18: "Four-dimensional" matrix space **M**. The "zero-dimensional" space **Z**.

Vector space inside a vector space

Ex : Subspace of R^3

A plane through origin



A line through origin

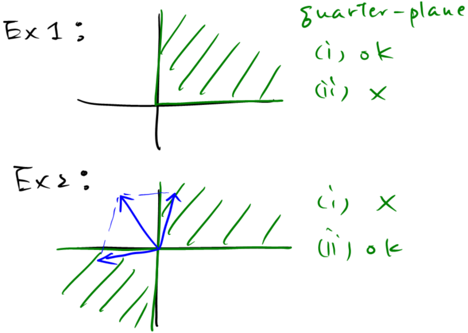
Def A subspace of a vector space is a set of vectors (including $\mathbf{0}$) that satisfies: $\exists \mathbf{v},\ \mathbf{w}$ in the subspace $\forall$ scalar c

(i) $\mathbf{v} + \mathbf{w}$ is in the subspace
(ii) $c\mathbf{v}$ is in the subspace
(closed under all lin. comb.)
(ADD & MUL follows from the host space
$\Rightarrow$ 8 rules are satisfied only need to worry about lin. comb.)

Session 9 Subspace

Fact Every subspace contains $\mathbf{0}$
(follows from (ii) with $c = 0$)

Not a subspace



Fact A subspace containing **v** **w** must contains all lim. comb. c**v** + d**w**

(smallest subspace containing **v** & **w** is the set of all comb.of **vw**)

Recall : P (Any plane through **0**)

L (Any line through **0**) are subspace in R^3

Q : Is $P \bigcup L$ a subspace?

No ! (i) fails

Q : Is $P \bigcap L$ a subspace?

Yes!

In general, for any subspacse S & T, $S \bigcap T$ is also a subspace

(If **v**,**w** on $S \bigcap T$,**v**+**w** in S , **v**+**w** in T

$\Rightarrow$ **v**+ **w** in $S \bigcap T$, similarity for c**v**)

Column space of A (Important subspace)

Def The col. space of A is the vector space made up of all possible lin. comb. of col.s of A notation : $C(A)$

$A\mathbf{x} = \mathbf{b}$

Q : Given a matrix A, for what vector $\mathbf{b}$ does $A\mathbf{x} = \mathbf{b}$ have a sol.?

Ex:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

(row picture)

4 eqns. 3 unknowns

$\Rightarrow A\mathbf{x} = \mathbf{b}$ does not have a sol. for every choice of $\mathbf{b} \Rightarrow \mathbf{b} \in C(A)$

Q : What are those **b** ?

Another perspective : only 3 col. vectors cannot fill the entire 4D space $\rightarrow$some **b** cannot be expressed as linear comb. of col.s of A

fact The system $A\mathbf{x} = \mathbf{b}$ is solvable iff **b** is in the col. space of A

When $\mathbf{b} \in \mathrm{C}(A)$, **b** is a lin. comb. of col.s of A

i.e., $\mathbf{b} = x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n$ for some $x_1 \cdots x_n$

the comb. give you sol. to $A\mathbf{x} = \mathbf{b}$

Back to Ex :

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 3 & 1 & 4 \\ 4 & 1 & 5 \end{bmatrix}$$

Q : What can we say about C(A)?

col. vectors lin. dependent or indep.?

or Does each col. contribute sth. new to the subspace ?

col.3 $=$ col.1 $+$ col.2 (lin. dependent)

(C(A) is 2D subspace of R^4)

In general,

$A_{m\times n}$: n col.s, each with m dim. $\Rightarrow$ C(A) is a subspace of R^m (not R^n)

Q : Is C(A) really a subspace ?

Yes! if $\mathbf{b}, \mathbf{b}^{'} \in C$(A),$\mathbf{b}$,$\mathbf{b}^{'}$ are comb. of col.s of $A \Rightarrow \mathbf{b}+\mathbf{b}^{'}$ still comb. of col.s of $A \Rightarrow c\mathbf{b}$ still comb. of col.s of A

(or $A\mathbf{x}=\mathbf{b}, A\mathbf{x}^{'}=\mathbf{b}^{'} \Rightarrow A(\mathbf{x}+\mathbf{x}^{'})=\mathbf{b}+\mathbf{b}^{'}\quad A(c\mathbf{x})=c\mathbf{b}$)

Recall : all comb. of **v**, **w** is the smallest subspace containing **v**, **w**

Notation : for a vector space V

S = set of vectors in V

SS = all comb. of vectors in S

(span of S : smallest subspace containing S)

Note:

The smallest possible col. space $A = \mathbf{0}$

(only contain $\mathbf{0}$)

The largest possible col. space R^m

(Ex : C(I) or any nonsingular m×m matrix)

(We can use elimination to solve it)

(This is more general than Ch.1. Now, we allow singular matrices rectangular matrices of any shape)