# EE205003 Session 8

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Elimination = Factorization: $A = LU$

$$A \xrightarrow[\text{steps}]{\text{Elimination}} U$$

or $EA = U \Rightarrow A = E^{-1}U = LU$
$(A \to E_{21}A \to E_{31}E_{21}A \to \cdots \to U)$

A 2x2 Ex

$$E_{21}A = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix} = U$$

$$\Rightarrow A = E_{21}^{-1}U = \underbrace{\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}}_{\text{L}} \underbrace{\begin{bmatrix} 2 & 1 \\ 0 & 5 \end{bmatrix}}_{\text{U}} = \begin{bmatrix} 2 & 1 \\ 6 & 8 \end{bmatrix} = A$$

If no row change, $3 \times 3$ Ex

$$(E_{32}E_{31}E_{21})A = U$$

$$\Rightarrow A = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}U$$

$$(= LU) - - -(1)$$

Note 1:

every inverse matrix $E_{21}^{-1}, E_{31}^{-1}, E_{32}^{-1}$ is lower triangular with off-diagonal entry $l_{ij}$ to undo $-l_{ij}$ for $E_{ij}$

Note 2:

Eqn. (1) shows

$$(E_{32}E_{31}E_{21})A = U \Rightarrow A = (E_{21}^{-1}E_{31}^{-1}E_{32}^{-1})U$$

Also lower - triangular $\qquad L$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ l_{21} & 1 & 0 \\ l_{31} & l_{32} & 1 \end{bmatrix} \text{(determined exactly by } l_{ij}\text{)}$$

Fact If no row change, $U$ has pivots on its diagonal, $L$ has all 1's on its diagonal $l_{ij}$ below the diagonal

Ex: $E_{31} = I$

$$\underbrace{\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\0 & -5 & 1\end{bmatrix}}_{E_{32}}\underbrace{\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\0 & 0 & 1\end{bmatrix}}_{E_{21}} = \underbrace{\begin{bmatrix}1 & 0 & 0\\-2 & 1 & 0\\10 & -5 & 1\end{bmatrix}}_{E}$$

(row2$^{new}$ =row2 - 2$\cdot$ row1)

row3 - 5$\cdot$row2$^{new}$

= row3 - 5$\cdot$(row2 - 2$\cdot$row1) (starting from top)

= row3 - 5$\cdot$row2 + 10$\cdot$row1

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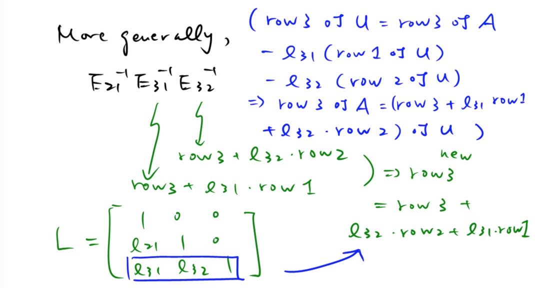
But $L = (E_{32}E_{31}E_{21})^{-1}U = E_{21}^{-1}E_{31}^{-1}E_{32}^{-1}$

$$= E_{32}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix}$$

(row3$^{new}$ =row3 + 5$\cdot$ row2)

(row2 + 2$\cdot$row1)(bottom up) (does NOT involve row3$^{new}$)



Factor out diagonal matrix

$$U = \begin{bmatrix} d_1 & & & \\ & d2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \begin{bmatrix} 1 & \frac{u_{12}}{d_1} & \frac{u_{13}}{d_1} & \cdots \\ & 1 & \frac{u_{23}}{d_2} & \vdots \\ & & \ddots & \vdots \\ & & & 1 \end{bmatrix}$$

$$\Rightarrow A = LDU$$

$$\left( \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 8 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix} \right)$$

Q: When do we use LU?

Most computer code use LU to solve $A\mathbf{x} = \mathbf{b}$

One square system $=$ Two triangular systems

Step1 : Factor $A = LU$ (get L for free)

Step2 : solve **b** using $L$

(Solve $L\mathbf{c} = \mathbf{b}$, then solve $U\mathbf{x} = \mathbf{c}$)

(forward & backward substitution)

$(L(U\mathbf{x}) = \mathbf{b} \Rightarrow \ A\mathbf{x} = \mathbf{b})$

Ex: $\begin{array}{r} u + 2v = 5 \\ 4u + 9v = 21 \end{array} \Rightarrow \begin{array}{r} u + 2v = 5 \\ v = 1 \end{array}$

or $A = \begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} \Rightarrow U = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, L = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix}$

$L\mathbf{c} = \mathbf{b} \Rightarrow \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \mathbf{c} = \begin{bmatrix} 5 \\ 21 \end{bmatrix} \Rightarrow \mathbf{c} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$U\mathbf{x} = \mathbf{c} \Rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\Rightarrow$back sub.$\Rightarrow \mathbf{x} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

Cost of Elimination

For a n×n matrix, to produce zeros below the first pivot

need $\sim n^2$ mul. & $u^2$ substraction

Eg. $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 3 \\ 0 & \times & \times \\ 0 & \times & \times \end{bmatrix}$

in fact $n(n-1)$

Next stage clears out $2^{nd}$ col. below $2^{nd}$ pivots $\sim (n-1)^2$ mul & sub.

$\vdots$

To reach $U$, need $\sim n^2 + (n-1)^2 + \cdots + 1^2$

$= \frac{1}{3}n(n+\frac{1}{2})(n+1) \cong \frac{1}{3}n^3$

Q: How about right side b ?

Step 1: substract multiples of $b_1$ from $b_2, \cdots, b_n$ $(n-1)$ mul & sub
Step 2: substract multiples of $b_2$ from $b_3, \cdots, b_n$ $(n-2)$ mul & sub

$$\vdots$$

$$(n-1) + (n-2) + \cdots + 1 + 1 + 2 + \cdots + n = n^2$$

Back substitution ↑ ↓

compute $x_n$ 1 ,, (small compared with $\frac{1}{3} n^3$)

,, $x_{n-1}$ 2 ,,

$$\vdots$$

Q: What if there are row exchanges ?

Use permutation matrix $P$

## Transposes & Permutation

### Transpose

$(A^T)_{ij} = A_{ji}$ (exchange row & col.)

Ex:

If $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 4 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 3 & 4 \end{bmatrix}$

(transpose of lower triangular is upper triangular)

### Rules

sum: $(A+B)^T = A^T + B^T$
product: $(AB)^T = B^T A^T$
inverse: $(A^{-1})^T = (A^T)^{-1}$

$(AB)^T = B^T A^T$

pf: Start with $A\mathbf{x} = \begin{bmatrix}\mathbf{a}_1 & \cdots & \mathbf{a}_n\end{bmatrix}\begin{bmatrix}x_1 \\ \cdots \\ x_n\end{bmatrix}$

$$= x_1\mathbf{a_1} + \cdots + x_n\mathbf{a}_n \quad \text{combine col. of } A$$

$$\Rightarrow (A\mathbf{x})^T = x_1\mathbf{a}_1^T + \cdots + x_n\mathbf{a}_n^T$$

$\mathbf{x}^T A^T = \begin{bmatrix}x_1 & \cdots & x_n\end{bmatrix}\begin{bmatrix}\mathbf{a}_1^T \\ \vdots \\ \mathbf{a}_n^T\end{bmatrix}$ combine row of $A^T$

$x_1^T\mathbf{a}_1 + \cdots + x_n^T\mathbf{a}_n \quad \Rightarrow (A\mathbf{x})^T = \mathbf{x}^T A^T$

For $B = \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}$

$$(AB)^T = \begin{bmatrix} A\mathbf{x}_1 & \cdots & A\mathbf{x}_n \end{bmatrix}^T = \begin{bmatrix} (A\mathbf{x}_1)^T \\ \vdots \\ (A\mathbf{x}_n)^T \end{bmatrix}$$

$$= \begin{bmatrix} \mathbf{x}_1^T A^T \\ \vdots \\ \mathbf{x}_n^T A^T \end{bmatrix} = B^T A^T$$

Can extend to 3 or more factors:

$(ABC)^T = C^T B^T A^T$

$(A^{-1})^T = (A^T)^{-1}$

pf: $AA^{-1} = I \Rightarrow (AA^{-1})^T = I^T = I$
$\Rightarrow (A^{-1})^T A^T = I \Rightarrow (A^{-1})^T = (A^T)^{-1}$(left inverse)

Similarly for right inverse $(A^{-1}A = I)$
$\Rightarrow$A$^T$ is invertible iff A is invertible

Symmetric matrix

$A^T = A$ or $a_{ji} = a_{ij}$

Ex: $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A^T$

Note: the inverse of a symmetric matrix is also symmetric

$((A^{-1})^T = (A^T)^{-1} = A^{-1}$ if $A$ symmetric)

Symmetric product

$R^T R$ is always symmetric for any R

$((R^T R)^T = R^T (R^T)^T = R^T R)$

(For symmetric $A$, $A = LDU \Rightarrow A = LDL^T$)

Permutation

Def A permutation matrix $P$ has the rows of the identity $I$ in any order

Ex: 3x3 permutation matrices

$$I = \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix}, P_{21} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix}, \mathrm{P}_{32}P_{21} = \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix}$$

$$P_{31} = \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix}, P_{32} = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix}, P_{21}P_{32} = \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}$$

there are n! permutation matrices of order n

Fact $P^{-1} = P^T$

$$P^T P = \begin{bmatrix} \mathbf{p}_1^T \\ \vdots \\ \mathbf{p}_n^T \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 & \cdots & \mathbf{p}_n \end{bmatrix} = I \Rightarrow P^{-1} = P^T$$

$$\because P_i^T P_j = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

Q: What if there are row exchanges ?

$PA = LU$

put all rows of $A$ in right order

If $A$ is invertible, $PA = LU$ s.t. $U$ has full sets of pivots

Ex:

$$\underset{A}{\begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \rightarrow \underset{PA}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 2 & 7 & 9 \end{bmatrix}} \rightarrow \underset{l_{31} = 2}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 7 \end{bmatrix}}$$

$$\rightarrow \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix} \Rightarrow PA = LU$$

$\uparrow$

$$l_{32} = 3 \quad \underset{P}{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} A = \underset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 4 \end{bmatrix}}$$