EE 205003 Session 6

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Addition

$A_{(m\times n)}$: m rows, n columns
$B_{(p\times q)}$: p rows, q columns

Q: Can you do $A+B$?

Only when $m=p, n=q$
$\Rightarrow$ two matrices are of same size

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}2 & 2\\4 & 4\end{bmatrix}=\begin{bmatrix}3 & 4\\7 & 8\end{bmatrix} \ (\checkmark)$$

$$\begin{bmatrix}1 & 2\\3 & 4\end{bmatrix}+\begin{bmatrix}2\\4\end{bmatrix} \ (\text{✗})$$

Multiplication

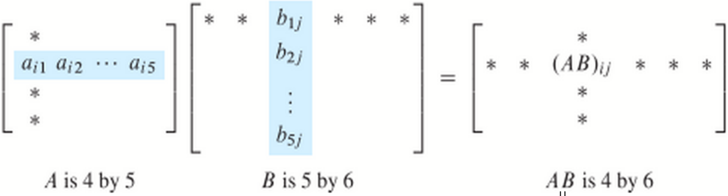
Q: Can you do AB?

If A has n col.s, we can do AB
only when B has n rows
$\Rightarrow A_{m\times n}B_{n\times p} = C_{m\times p}$
[check of dim. is important to trace errors]

Four different ways of thinking $AB = C$

Standard (rows × col.s) inner product

$$(AB)_{ij} = (\text{row } i \text{ of } A) \cdot (\text{col } j \text{ of } B)$$



$\| \; C$

or $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

Ex1

$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 2 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 1 & 0 \end{bmatrix}$$

(dim. chk : square matrices can be multiplied
iff they are of same size)

(if $n \times n$: involves n^2 dot products
each dot product = n multiplications
$\Rightarrow$ total n^3 multiplications)

Ex2

$$\begin{bmatrix} 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 + 6 = 8 \text{ (inner product)}$$

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2 \\ 0 & 2 & 4 \\ 0 & 3 & 6 \end{bmatrix} \text{ (outer product)}$$

Columns

$$C = AB = A\left[\mathbf{b}_1, \quad \mathbf{b}_2, \quad \cdots, \quad \mathbf{b}_p\right] = \left[A\mathbf{b}_1, \quad A\mathbf{b}_2, \quad \cdots, \quad A\mathbf{b}_p\right]$$

each col. of C is $A\mathbf{b}_i$ (lin. comb. of col.s of A)

$\Rightarrow$ each col. of C is a lin. comb. of col.s of A

Rows

$$\begin{bmatrix} \mathbf{c}_1^\mathsf{T} \\ \mathbf{c}_2^\mathsf{T} \\ \vdots \\ \mathbf{c}_m^\mathsf{T} \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1^\mathsf{T} \\ \mathbf{a}_2^\mathsf{T} \\ \vdots \\ \mathbf{a}_m^\mathsf{T} \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1^\mathsf{T} B \\ \mathbf{a}_2^\mathsf{T} B \\ \vdots \\ \mathbf{a}_m^\mathsf{T} B \end{bmatrix}$$

each row of C is $\mathbf{a}_i B$ (lin. comb. of rows of B)

$\Rightarrow$ each row of C is a lin. comb. of rows of B

Column times row

each is a $m \times p$ matrix

$$[\mathbf{a}_1 \mathbf{a}_2 \cdots \mathbf{a}_n] \begin{bmatrix} \mathbf{b}_1^\intercal \\ \mathbf{b}_2^\intercal \\ \vdots \\ \mathbf{b}_n^\intercal \end{bmatrix} = \mathbf{a}_1 \mathbf{b}_1^\intercal + \mathbf{a}_2 \mathbf{b}_2^\intercal + \cdots + \mathbf{a}_n \mathbf{b}_n^\intercal$$

Ex

$$\begin{bmatrix} 1 & 4 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 3 & 2 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \begin{bmatrix} 3 & 2 \end{bmatrix} + \begin{bmatrix} 4 \\ 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \end{bmatrix}$$

$$\| $$

$$\begin{bmatrix} 3 & 2 \\ 6 & 4 \end{bmatrix}$$

(3, 2) lies in the same line as (6, 4)

$\Rightarrow$ row space is a line

Similarly, col. space is also a line

Blocks

$$\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix} \begin{bmatrix} B_1 & B_2 \\ B_3 & B_4 \end{bmatrix} = \begin{bmatrix} C_1 & C_2 \\ C_3 & C_4 \end{bmatrix}$$

here $C_1 = A_1 B_1 + A_2 B_3$

Ex Elimination by blocks

$$A = \begin{bmatrix} 1 & \times & \times \\ 3 & \times & \times \\ 4 & \times & \times \end{bmatrix}$$

one at a time

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \; E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

Ex Elimination by blocks (cont.)

$$E = E_{21}E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ -4 & 0 & 1 \end{bmatrix}$$

$$EA = \left[\begin{array}{c|cc} 1 & 0 & 0 \\ \hline -3 & 1 & 0 \\ -4 & 0 & 1 \end{array}\right] \left[\begin{array}{c|cc} 1 & \times & \times \\ \hline 3 & \times & \times \\ 4 & \times & \times \end{array}\right] = \left[\begin{array}{c|cc} 1 & \times & \times \\ \hline 0 & \times & \times \\ 0 & \times & \times \end{array}\right]$$

$$\left(\begin{bmatrix} -3 \\ -4 \end{bmatrix} 1 + I \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right)$$ (Schur component)

$$\left(\left[\begin{array}{c|c} I & 0 \\ \hline -CA^{-1} & I \end{array}\right] \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] = \left[\begin{array}{c|c} A & B \\ \hline 0 & D - CA^{-1}B \end{array}\right]\right)$$

(check : $-CA^{-1}A + C = 0, -CA^{-1}B + D$)

The Laws for matrix operations

For addition

$$A + B = B + A \qquad \text{commtative}$$
$$c(A + B) = cB + cA \qquad \text{distributive}$$
$$A + (B + C) = (A + B) + C \qquad \text{associative}$$

For multiplication

$$AB \neq BA \qquad \text{commutative broken !}$$
$$C(A + B) = CA + CB \qquad \text{distributive from left}$$
$$(A + B)C = AC + BC \qquad \text{distributive from right}$$
$$A(BC) = (AB)C \qquad \text{associative}$$

$\underline{AB \neq BA}$

Obvious if A, B not square

$A_{(m\times n)}B_{(n\times p)} = AB(m \times p)$

$B_{(n\times p)}A_{(m\times n)}$ (not legal if $p \neq m$)

$B_{(n\times m)}A_{(m\times n)} = BA_{(n\times n)}$ $(p = m)$

$(AB_{(m\times m)} \neq BA$ if $m \neq n)$

Even if both square,

$$AB = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Exception

$AI = IA$ (only cI commutes with other matrices)

$A(B + C) = AB + AC$

$A(\mathbf{b} + \mathbf{c}) = A\mathbf{b} + A\mathbf{c}$ (can prove a col. at a time)

Powers

$A^p = AA \cdots A$

$(A^p)(A^q) = A^{p+q}$, $(A^p)^q = A^{pq}$