EE 203001 Session 5

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Elimination using matrices

$$2x_1 + 4x_2 - 2x_3 = 2$$
$$4x_1 + 9x_2 - 3x_3 = 8$$
$$-2x_1 - 3x_2 + 7x_3 = 10$$

In matrix form

$$\underbrace{\begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{\mathbf{x}} \underset{=}{=} \underbrace{\begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}}_{\mathbf{b}}$$

(Sol. is x=(-1,2,2))

Column form:

$$A\mathbf{x} = (-1)\begin{bmatrix} 2 \\ 4 \\ -2 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 9 \\ -3 \end{bmatrix} + 2\begin{bmatrix} -2 \\ -3 \\ 7 \end{bmatrix} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix}$$

In general

$$A\mathbf{x} = x_1(col.1) + \cdots + x_n(col.n)$$

Row form

the i-th component of $A\mathbf{x}$ = dot product of the i-th row of A

$$\begin{bmatrix} a_{i1} & a_{i2} & \cdots & a_{in} \end{bmatrix} \text{ with } \mathbf{x}$$

$$= \sum_{j=1}^{n} a_{ij} x_j$$

The matrix form of one elimination step

Recall: the 1st step of Elimination

(equ.2) - 2$\times$(equ.1)

Focus on the right side of Ax=b

$$\mathbf{b} = \begin{bmatrix} 2 \\ 8 \\ 10 \end{bmatrix} \Rightarrow \mathbf{b}_{new} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}$$

Q: Can we represent this step using a matrix?

Yes! Elimination matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 - 2b_1 \\ b_3 \end{bmatrix}$$

1st & 3rd row of identity matrix

⇒ Row 1 & 3 of b stay the same

Q: How to construct a Elimination matrix E_{ij}?

Use an identity matrix I. E_{ij} that substracts a multiple l of row j from row i has the extra nonzero entry $-l$ in the i,j position

Ex:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l & 0 & 1 \end{bmatrix}, l = -1$$

$$I\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 10 \end{bmatrix}, E_{31}\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \mathbf{b} = \begin{bmatrix} 2 \\ 4 \\ 12 \end{bmatrix}$$

Q: How about the left side of $A\mathbf{x} = \mathbf{b}$

The purpose of E_{31} is to produce a zero in the (3,1) position of the matrix

Elimination using matrices

Apply E's to produce zeros below the pivot

Q: What is the first E?

$E_{21} \rightarrow E_{31} \rightarrow E_{32}$

Note: the vector x stays the same but coeff. matrix is changed

Start with :

$$A\mathbf{x} = \mathbf{b}$$

multiply by E:

$$EA\mathbf{x} = E\mathbf{b}$$

Q: How do we multiply two matrices?

We expect E acting on A : subtracts 2×(row1) from (row2) of A

$$EA = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

Note: $A\mathbf{x}=\mathbf{b}$

$E(A\mathbf{x})=E\mathbf{b}$

same as $(EA)\mathbf{x}=E\mathbf{b}$

For matrices,

Associative law is true, i.e.,

$$A(BC)=(AB)C$$

Commutative law is false, i.e.,

often $AB \neq BA$

Another requirement for matrix multiplication

If B has only one $col(\mathbf{b})$

then EB should agree with $E\mathbf{b}$

In fact, if $B=\begin{bmatrix}\mathbf{b_1} & \mathbf{b_2} & \mathbf{b_3}\end{bmatrix}$

$\Rightarrow EB=\begin{bmatrix}E\mathbf{b_1} & E\mathbf{b_2} & E\mathbf{b_3}\end{bmatrix}$

$$EB = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 4 & 9 & -3 \\ -2 & -3 & 7 \end{bmatrix} = \begin{bmatrix} 2 & 4 & -2 \\ 0 & 1 & 1 \\ -2 & -3 & 7 \end{bmatrix}$$

P_{ij} **for a row exchange (permutation matrix)**

$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$ **exchange component 2&3 for any vector**

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix}$$

So it also exchanges row 2&3 for any matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2 & 4 & 1 \\ 0 & 0 & 3 \\ 0 & 6 & 5 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 1 \\ 0 & 6 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

In general

P_{ij} **is the identity matrix with row i&j exchanged**

The augmented matrix

Elimination does same row operations to A & to b $\Rightarrow$ We can include b as an extra column.

$$[A \quad b] = \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$E_{21}\,[A \quad b] = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 & 2 \\ 4 & 9 & -3 & 8 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 & -2 & 2 \\ 0 & 1 & 1 & 4 \\ -2 & -3 & 7 & 10 \end{bmatrix}$$

$$x_2 + x_3 = 4$$

By rows

Each row of E acts on [A b] to give a row of [EA Eb]

By columns

E acts of each column of [A b] to give a column of [EA Eb]

Step by step

$A \rightarrow E_{21}A \rightarrow E_{31}E_{21}A \rightarrow E_{32}E_{31}E_{21}A$

P.61: 2.3A